

APPH 4200

Physics of Fluids

Boundary Layers (Ch. 10)

Answer: A pioneer in the mathematical development of aerodynamics who conceived the idea of a fluid “boundary layer”, considered by many as “the greatest single discovery in fluid dynamics.”

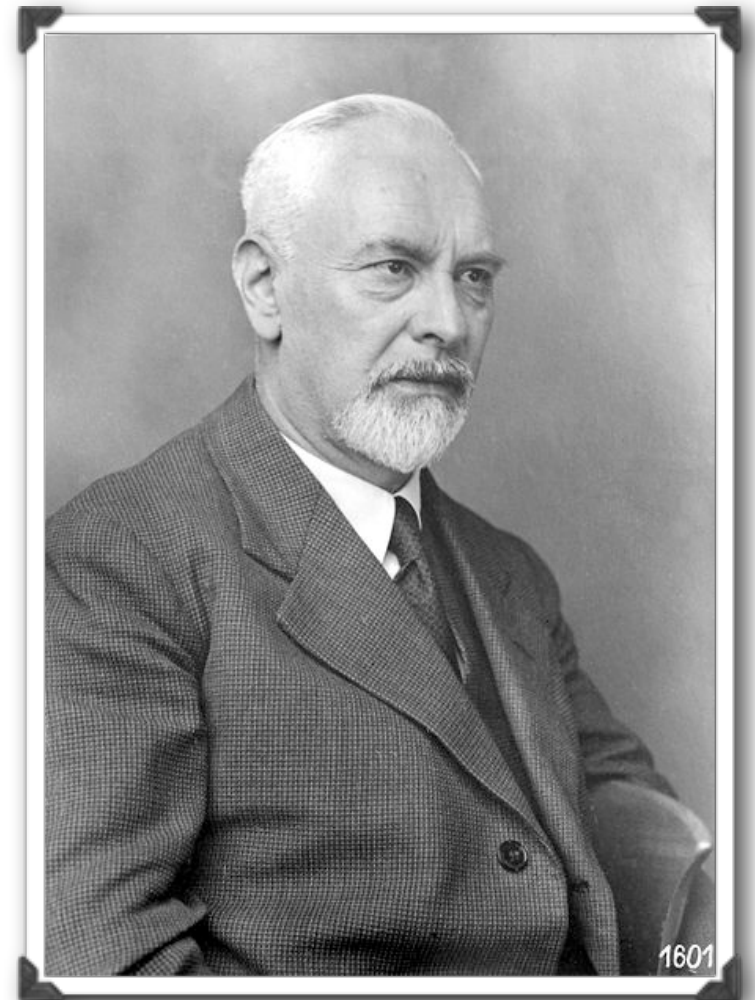
Question: Who was Ludwig Prandtl?

Ludwig Prandtl

During the week of 8 August 1904, a small group of mathematicians and scientists gathered in picturesque Heidelberg, Germany, known for its baroque architecture, cobblestone streets, and castle ruins that looked as if they were still protecting the old city. Home to Germany's oldest university, which was founded in 1386, Heidelberg was a natural venue for the Third International Mathematics Congress.

One of the presenters at the congress was Ludwig Prandtl, a 29-year-old professor at the Technische Hochschule (equivalent to a US technical university) in Hanover. Prandtl's presentation was only 10 minutes long, but that was all the time needed to describe a new concept that would revolutionize the understanding and analysis of fluid dynamics. His presentation, and the subsequent paper that was published in the congress's proceedings one year later, introduced the concept of the boundary layer in a fluid flow over a surface. In 2005, concurrent with the World Year of Physics celebration of, among other things, Albert Einstein and his famous papers of 1905, we should also celebrate the 100th anniversary of Prandtl's seminal paper. The modern world of aerodynamics and fluid dynamics is still dominated by Prandtl's idea. By every right, his boundary-layer concept was worthy of the Nobel Prize. He never received it, however; some say the Nobel Committee was reluctant to award the prize for accomplishments in classical physics.

Physics Today, Dec 2005



Feb 4, 1875 – Aug 15, 1953
German Scientist

A very satisfactory explanation of the physical process in the boundary layer [*Grenzschicht*] between a fluid and a solid body could be obtained by the hypothesis of an adhesion of the fluid to the walls, that is, by the hypothesis of a zero relative velocity between fluid and wall. If the viscosity was very small and the fluid path along the wall not too long, the fluid velocity ought to resume its normal value at a very short distance from the wall. In the thin transition layer [*Übergangsschicht*] however, the sharp changes of velocity, even with small coefficient of friction, produce marked results.

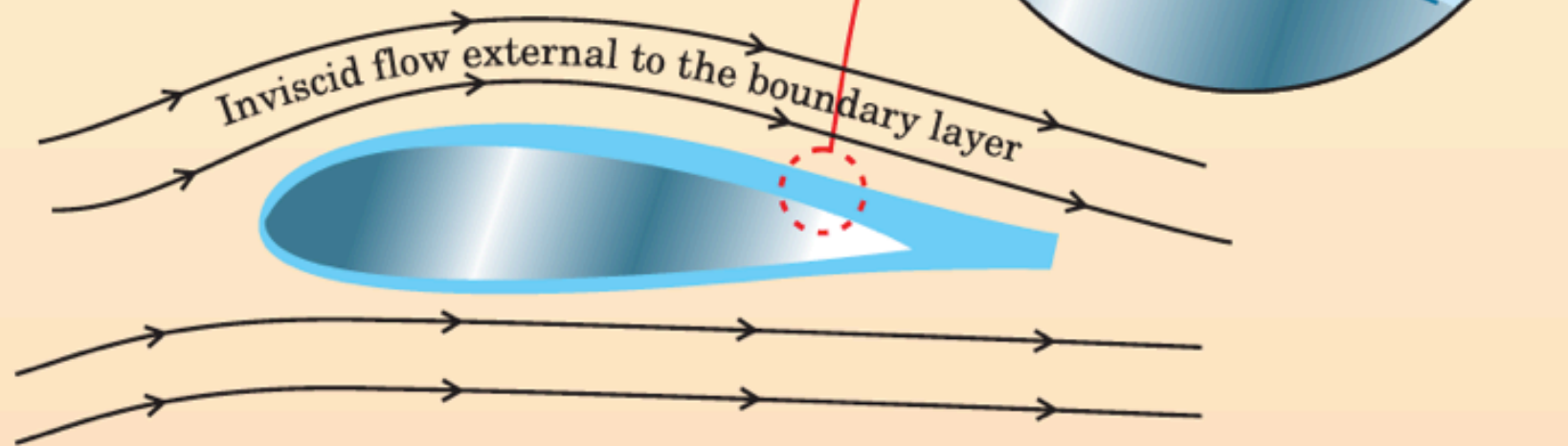
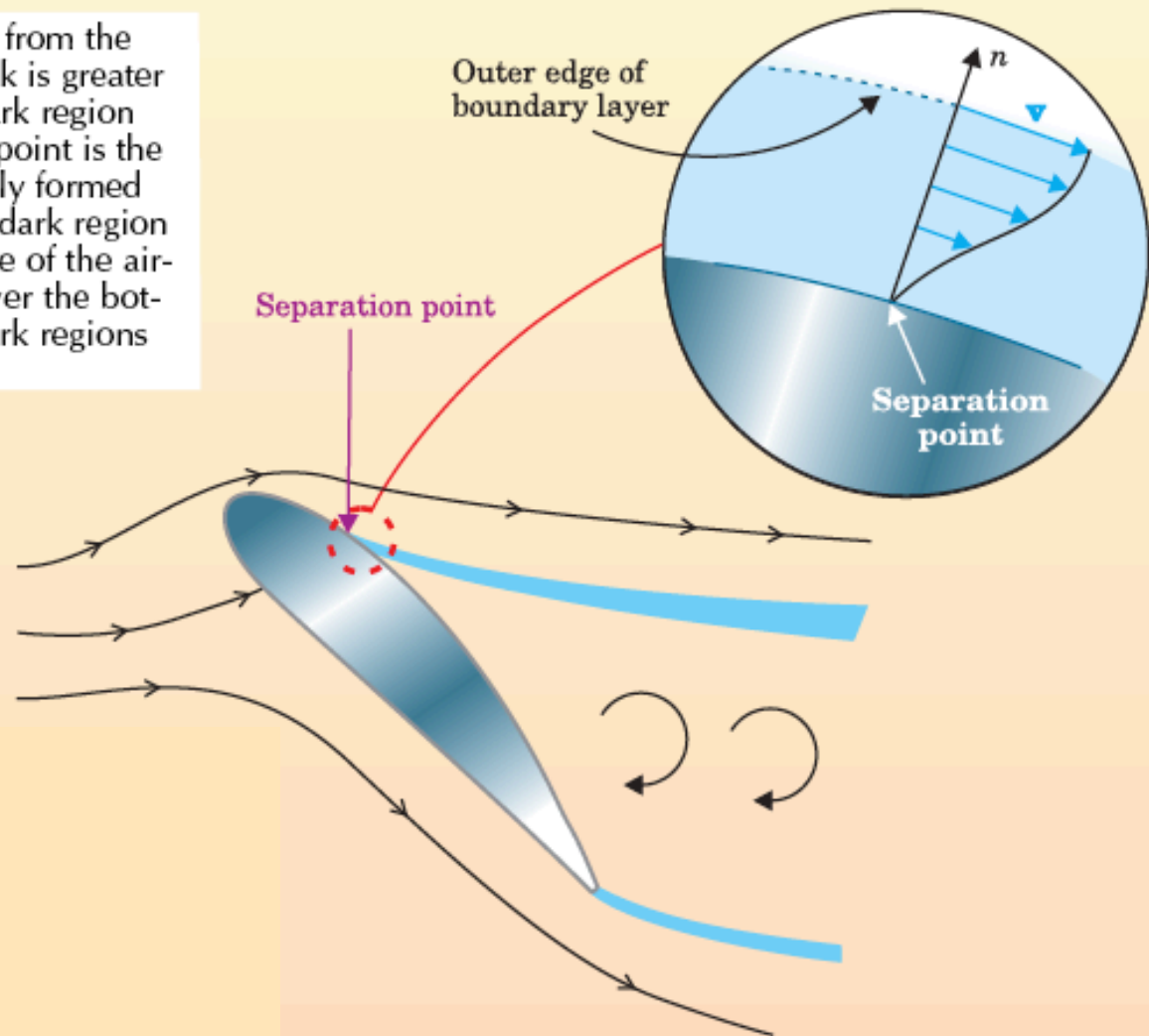


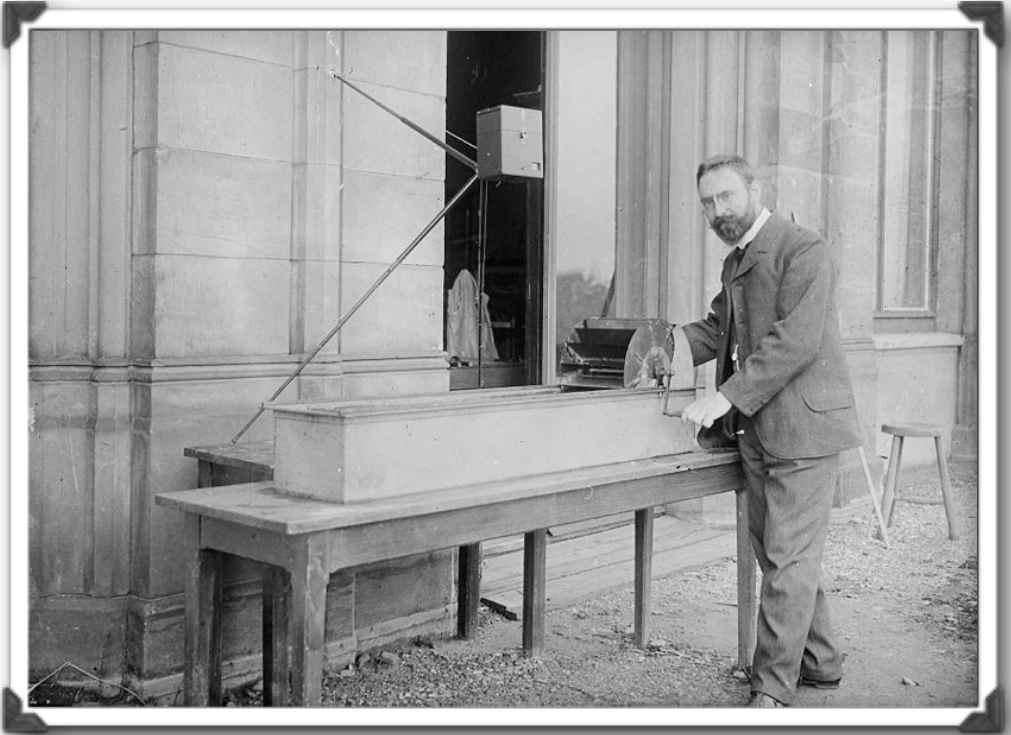
Figure 2. A fluid flow may be viewed as comprising two parts. In a thin boundary layer (blue) adjacent to the surface, the effects of friction are dominant. Outside the boundary layer, the flow is inviscid. The blowup of the boundary layer shows how the flow velocity v changes, as a function of the normal distance n , from zero at the surface to the full inviscid-flow value at the outer edge.

In given cases in certain points fully determined by external conditions, the fluid flow ought to separate from the wall. That is, there ought to be a layer of fluid which, having been set in rotation by the friction on the wall, insinuates itself into the free fluid, transforming completely the motion of the latter, and therefore playing there the same part as the Helmholtz surfaces of discontinuity.

Figure 3. The boundary layer can separate from the top surface of an airfoil if the angle of attack is greater than the so-called stall angle. The upper dark region that trails downstream from the separation point is the remnant of the boundary layer that originally formed on the top surface of the airfoil. The lower dark region that trails downstream from the trailing edge of the airfoil is the remnant of the boundary layer over the bottom surface. When separated, these two dark regions are called shear layers, and they form the upper and lower boundaries of the separated flow region. Between the shear layers is a dead-air region. Due to the considerable flow separation illustrated here, the lift of the airfoil is dramatically reduced—the airfoil is stalled. The blowup shows the flow's velocity profile above the separation point.



Since the mid-1920s, work aimed at advancing, extending, and applying boundary-layer theory has increased exponentially. Such work has created lifetime careers for a large number of fluid dynamicists and aerodynamicists. The first serious industrial application of boundary-layer theory occurred in the late 1920s when designers began to use the theory's results to predict skin-friction drag on airships and airplanes. Prior to that time, they had been limited to using empirical data obtained primarily from wind tunnels. Such data usually were for the total drag, and the effect of skin friction was difficult to cull out. Furthermore, until the late 1920s, wind-tunnel data were notoriously inaccurate and the designers, conservative by nature, were reluctant to hinge their designs on them. But since the late 1920s, when the accuracy and value of skin-friction formulas obtained from boundary-layer theory became more appreciated, the results of the theory have become a standard tool of the airplane designer.



Prandtl's boundary-layer idea revolutionized how scientists conceptualized fluid dynamics. Before Prandtl, there was much confusion about the role of viscosity in a fluid flow. After Prandtl's paper, the picture was made clear; in most cases, viscosity only played a role in the thin layer of flow immediately adjacent to a surface. What a breakthrough in the analysis and understanding of a viscous flow! Before Prandtl, there was no mathematically based, quantitative means to calculate the drag due to friction on a surface immersed in a fluid flow. After Prandtl's paper, the fluid dynamicist could quantitatively calculate the skin-friction drag.

Extensions of Prandtl's work

If Prandtl had presented his paper in our electronic age of almost instant information dissemination, his boundary-layer concept would quickly have spread throughout the aerodynamics community. But at the turn of the century, information flowed much more slowly. Also, the Third International Mathematics Congress was an obscure setting for such an important contribution, and Prandtl's idea went virtually unnoticed by anybody outside of Göttingen for several years. It surfaced again in 1908 when Prandtl's student, Heinrich Blasius, published in the respected journal *Zeitschrift für Mathematik und Physik*, his paper "Boundary Layers in Fluids with Little Friction," which discussed 2D boundary-layer flows over a flat plate and a circular cylinder.⁵

$$u(\partial u / \partial x) + v(\partial u / \partial y) = \nu(\partial^2 u / \partial y^2) \quad (1)$$

$$\partial u / \partial x + \partial v / \partial y = 0 \quad (2)$$

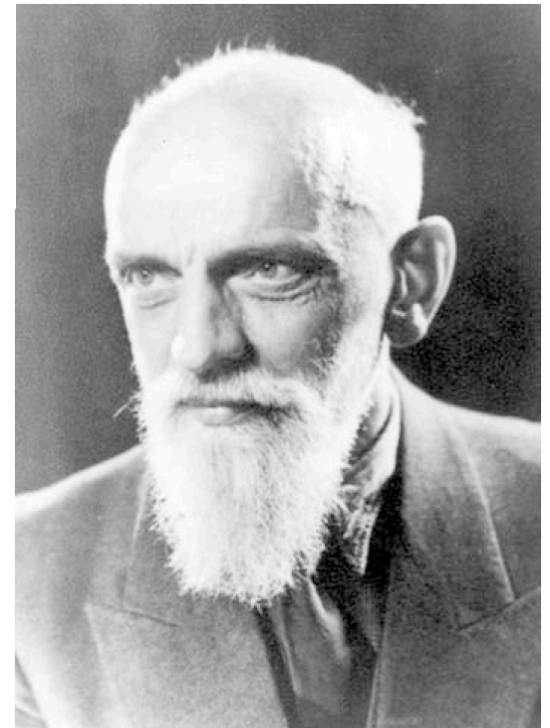
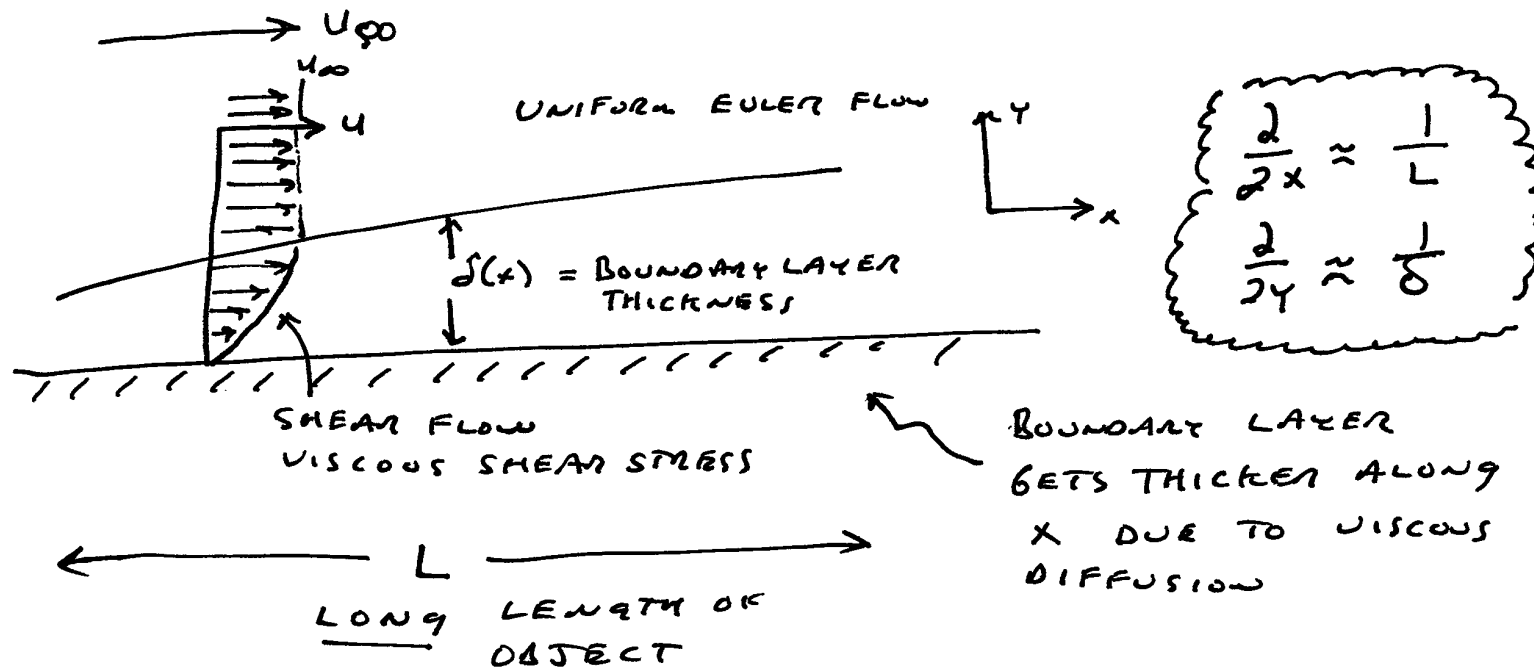


Fig. 1. Blasius in 1962, after retiring

Uniform Flow Across a Stationary Flat Plate

(Blasius, 1908)



$$\frac{\partial}{\partial x} \approx \frac{1}{L}$$

$$\frac{\partial}{\partial y} \approx \frac{1}{\delta}$$

$$\delta(x) \ll L$$

- STEADY FLOW
- INCOMPRESSIBLE
- VISCOUS ONLY WITHIN THIN BOUNDARY LAYER

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \bar{u}$$

$$\nabla \cdot \bar{u} = 0$$

OUTSIDE BOUNDARY:

$$\bar{u} \cdot \nabla \bar{u} = -\frac{1}{\rho} \nabla p \quad \text{or} \quad \nabla \left(\frac{1}{2} \rho \bar{u}^2 + p \right) = 0$$

ONLY INSIDE



How big is the boundary layer?

$$\bar{u} \cdot \nabla \bar{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \bar{u}$$

$$\hat{x}: u_x \frac{\partial u_x}{\partial x} \approx \nu \frac{\partial^2 u_x}{\partial y^2}$$

BECAUSE

$$(u_y) \ll (u_x) \quad \left| \frac{\partial}{\partial y} \right| \gg \left| \frac{\partial}{\partial x} \right|$$

SCALE: $\frac{u_\infty^2}{L} \sim \frac{\nu u_\infty}{\delta^2}$

$$\therefore \delta^2 = \frac{\nu L}{u_\infty} = L^2 \left(\frac{\nu}{L u_\infty} \right) = \frac{L^2}{Re}$$

VISCOUS DIFFUSION:

$$\delta(x) \sim \sqrt{\nu t}$$

HOW VISCOUS SCALE
LENGTH CHANGES IN
TIME: VISCOUS MOMENTUM
DIFFUSION!

BUT $t \sim L/u_\infty$ SO $\delta(x) \sim \sqrt{\nu L/u_\infty}$

$Re = \text{REYNOLDS NUMBER}$

$$= \frac{u_\infty L}{\nu}$$

What are the magnitudes of the terms in Navier-Stokes?

$$\bullet \quad u_y \sim u_\infty \frac{\delta}{L} \sim \frac{u_\infty}{\sqrt{Re}}$$

$$\bullet \quad |P - P_\infty| \sim \rho u_\infty^2 \quad \text{FROM BERNOULLI'S PRINCIPLE}$$

THEREFORE

$$\hat{x}: \quad u_x \frac{2u_x}{2x} + u_y \frac{2u_y}{2y} = -\frac{1}{\rho} \frac{2P}{2x} + \nu \frac{2^2 u_x}{2x^2} + \nu \frac{2^2 u_x}{2y^2}$$

$$\hat{y}: \quad u_x \frac{2u_y}{2x} + u_y \frac{2u_y}{2y} = -\frac{1}{\rho} \frac{2P}{2y} + \nu \frac{2^2 u_y}{2x^2} + \nu \frac{2^2 u_y}{2y^2}$$

$$\hat{x}: \quad u_x \frac{2u_x}{2x} + u_y \frac{2u_x}{2y} \approx -\frac{1}{\rho} \frac{2P}{2x} + \nu \frac{2^2 u_x}{2y^2}$$

$$\hat{y}: \quad 0 \approx -\frac{1}{\rho} \frac{2P}{2y} \quad \leftarrow$$

Blasius Flat Plate Solution

- SINCE FOR UNIFORM FLOW ALONG A FLAT PLATE $U_\infty = \text{CONSTANT}$, THEN $P = \text{CONSTANT ALONG } x$

- EQUATIONS TO SOLVE

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

- INCOMPRESSIBLE

$$(\nabla \cdot \bar{u} = 0)$$

$$\bar{u} = -\hat{z} \times \nabla \psi = \begin{cases} \hat{x} \frac{\partial \psi}{\partial y} \\ -\hat{y} \frac{\partial \psi}{\partial x} \end{cases}$$

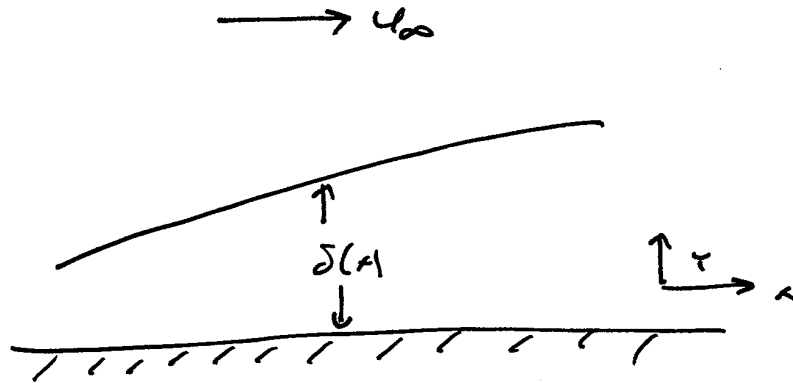
$$u_x = \frac{\partial \psi}{\partial y}$$

$$u_y = -\frac{\partial \psi}{\partial x}$$

- PDE NONLINEAR 2D

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

Boundary Conditions



$$\frac{\partial^4 u}{\partial y^4} - \frac{\partial^4 u}{\partial x^2 \partial y^2} = \gamma \frac{\partial^3 u}{\partial y^3}$$

- For $y \gg \delta$, $u \rightarrow u_\infty$ $\frac{\partial^4 u}{\partial y^4} \rightarrow 0$ as $y \rightarrow \infty$
- For $y = 0$, $u = 0$ ($u_x = u_y = 0$) $\frac{\partial^4 u}{\partial y^4} = \frac{\partial^4 u}{\partial x^2} = 0$ as $y = 0$

BLASIUS SOLUTION

SIMILARITY: $u_x(x, y) = u_\infty g(\eta)$

$$\eta \equiv \frac{y}{\delta(x)}$$

GENERAL SHAPE OF BOUNDARY DOES NOT CHANGE
ALONG BOUNDARY!

Finding the self-similar Stream function...

$$\begin{aligned}
 u_x &= \frac{2\psi}{2\gamma} & \text{THEN } \psi(x, \gamma) &= \int_0^\gamma d\gamma u_x \\
 &= u_\infty g(\eta) & &= \delta \int_0^\eta u_x d\eta \\
 \eta &\equiv \frac{\gamma}{\delta(x)} & &= \delta u_\infty \int_0^\eta g(\eta) d\eta
 \end{aligned}$$

$$\text{DEFINIS } g(\eta) \equiv \frac{df}{d\eta}$$

THEN

$$\psi(x, \gamma) = \delta u_\infty f(\eta)$$

WHAT IS $f(\eta)$?

$$\frac{2\psi}{2x} \frac{2^2\psi}{2x2\gamma} - \frac{2\psi}{2x} \frac{2^2\psi}{2\gamma^2} = \gamma \frac{2^3\psi}{2\gamma^3}$$

$$\left\{ \begin{aligned}
 \frac{2\psi}{2x} &= u_\infty \left[\frac{d\delta}{dx} f + \delta \frac{2f}{2x} \right] \\
 &= u_\infty \frac{d\delta}{dx} \left[f - \frac{df}{d\eta} \eta \right] \\
 \frac{2\psi}{2\gamma} &= u_\infty \frac{df}{d\eta} \quad \text{ETC}
 \end{aligned} \right.$$

←
SUBSTITUTE

What is $f(\eta)$?

$$\frac{24}{2\tau} \frac{2^2 4}{2 \times 2\tau} - \frac{24}{2\tau} \frac{2^2 4}{2\tau^2} = \gamma \frac{2^3 4}{2\tau^3}$$

$$-U_\infty^2 \frac{1}{\delta} \frac{\partial \delta}{\partial x} \eta f' f'' - U_\infty^2 \frac{1}{\delta} \frac{\partial \delta}{\partial x} f'' [f - f' \eta] = U_\infty \frac{\gamma}{\delta^2} f'''$$

OR

$$- \left(\frac{U_\infty \delta}{\gamma} \frac{\partial \delta}{\partial x} \right) f f'' = f'''$$

BUT $\frac{U_\infty \delta}{\gamma} \frac{\partial \delta}{\partial x} = \frac{1}{2}$ OR $\delta = \sqrt{\frac{\gamma x}{U_\infty}}$

THUS $\frac{1}{2} f f'' + f''' = 0$

$$\frac{df}{d\eta} \rightarrow 1 \text{ as } \eta \rightarrow \infty \quad (\tau \rightarrow \infty)$$

$$f(0) = \left. \frac{df}{d\eta} \right|_0 = 0 \text{ as } \tau \rightarrow 0 \quad \eta \rightarrow 0$$

Describing the boundary layer...

With $f(\eta)$ known, the boundary layer description is complete...

$$\psi(x, y) = U_\infty \delta f(\eta)$$

$$u_x = \frac{2y}{2\tau} = U_\infty \frac{df}{d\eta}$$

$$\eta = \frac{y}{\delta(x)}$$

$$u_y = -\frac{2y}{2x} = U_\infty \frac{d\delta}{dx} \left[f - \eta \frac{df}{d\eta} \right]$$

$$\frac{d\delta}{dx} = \frac{1}{2} \left(\frac{y}{U_\infty \delta} \right) = \frac{1}{2\sqrt{Re(x)}}$$

SHEAR STRESS AT WALL

$$\tau(y=0) = \rho \nu \frac{2u_x}{2\tau} = \rho \nu \frac{2^2 y}{2\tau^2} = \rho \nu U_\infty \frac{f''(0)}{\delta}$$

$$= \frac{0.332 \rho U_\infty^2}{\sqrt{Re(x)}}$$

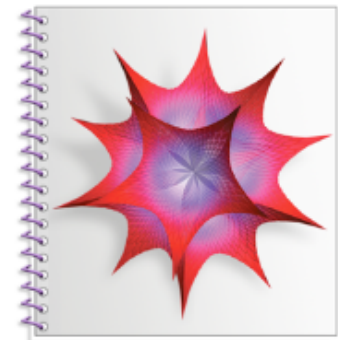
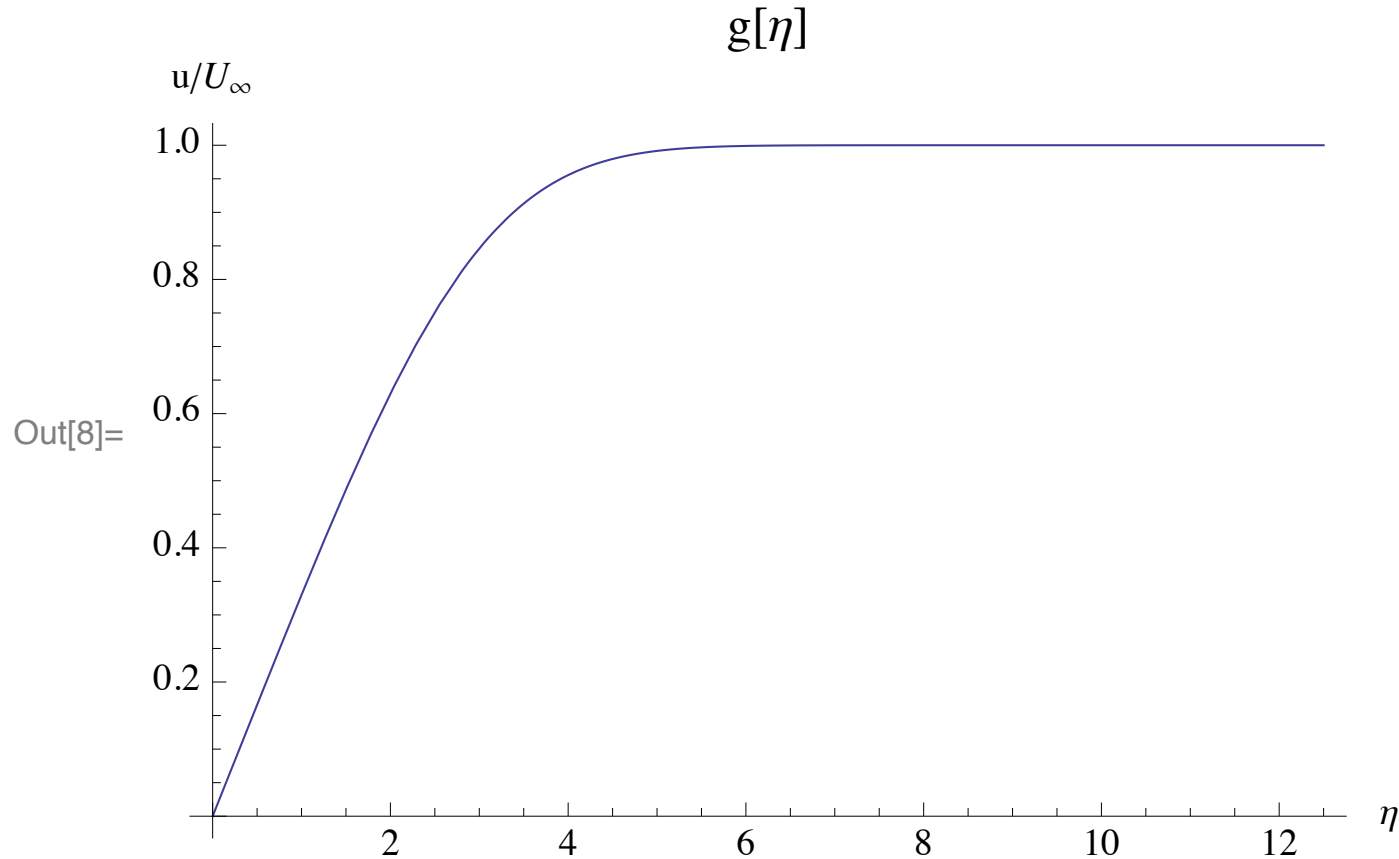
$$Re(x) \equiv \frac{U_\infty x}{\nu}$$

$$\text{COEFFICIENT OF DRAG} \sim \frac{\tau(x=0)}{\frac{1}{2} \rho U_\infty^2} = \frac{0.664}{\sqrt{Re(x)}}$$

$$\text{STOKES FLOW } C_D \sim \frac{24}{Re}$$

Numerical Solution

```
In[8]:= Plot[gBlasius[ $\eta$ ], { $\eta$ , 0,  $\eta$ Big / 2},  
PlotLabel  $\rightarrow$  "g[ $\eta$ ]", PlotRange  $\rightarrow$  All, AxesLabel  $\rightarrow$  {" $\eta$ ", "u/U $_{\infty}$ "}]
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Blasius_Boundary.nb

Long Flat Plates: Transition to Turbulence

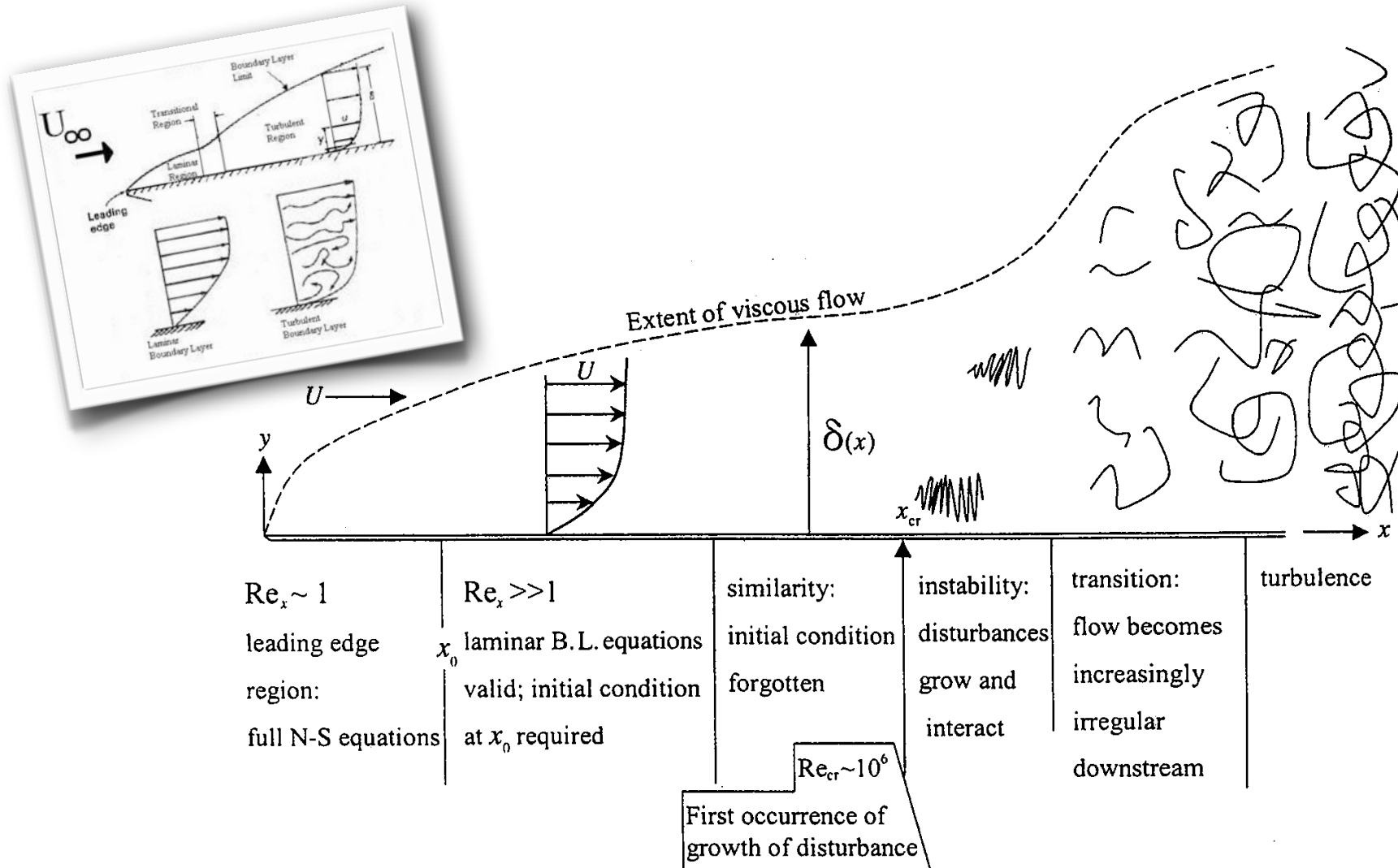


Figure 10.11 Schematic depiction of flow over a semiinfinite flat plate.

Long Flat Plates: Transition to Turbulence

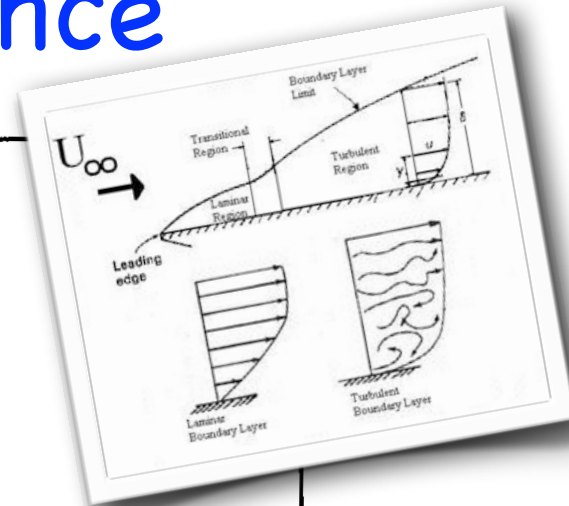
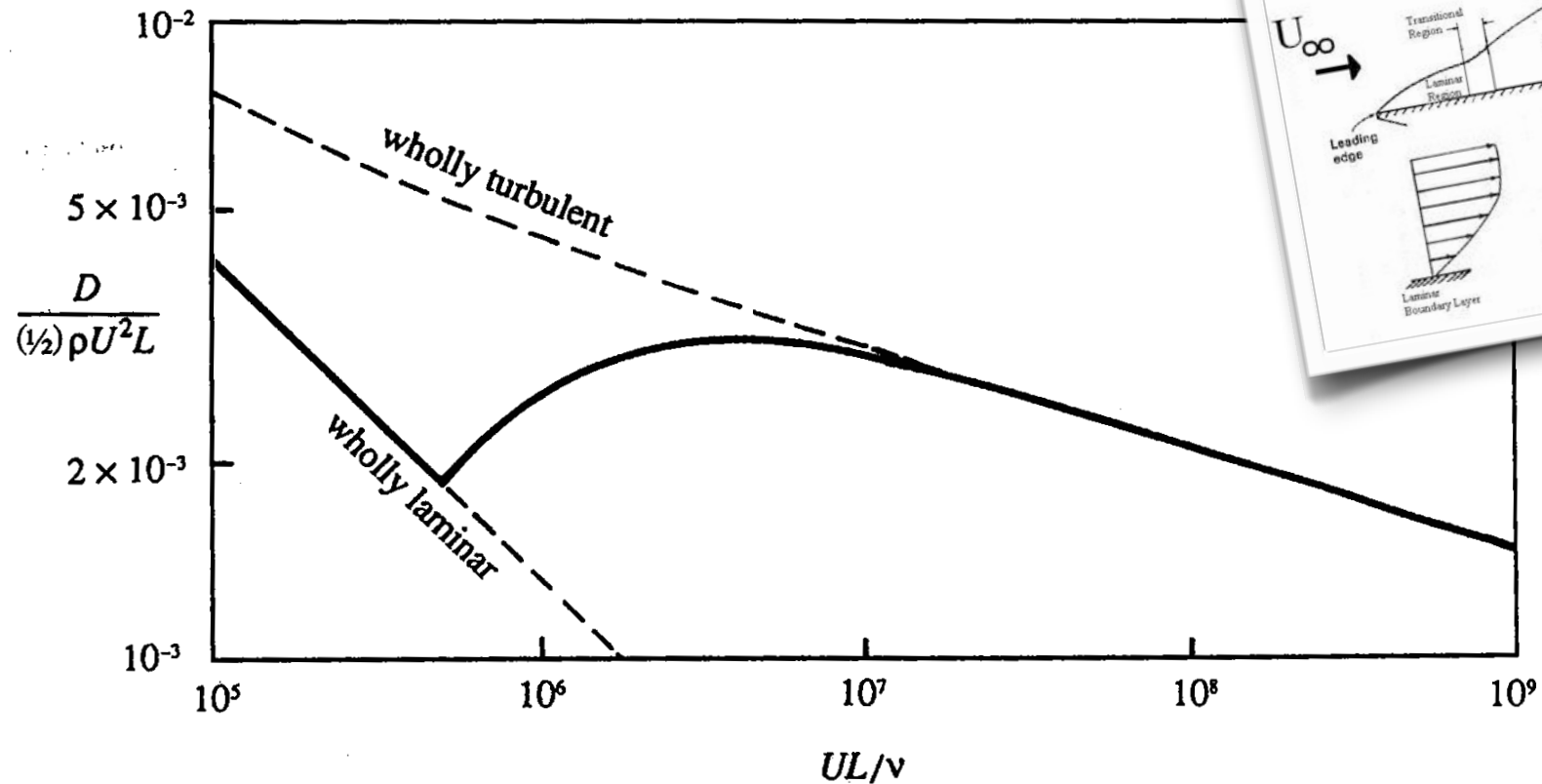


Figure 10.12 Measured drag coefficient for a boundary layer over a flat plate. The continuous line shows the drag coefficient for a plate on which the flow is partly laminar and partly turbulent, with the transition taking place at a position where the local Reynolds number is 5×10^5 . The dashed lines show the behavior if the boundary layer was either completely laminar or completely turbulent over the entire length of the plate.

On-Line Video

Fluid Mechanics (Boundary Layers part 1)

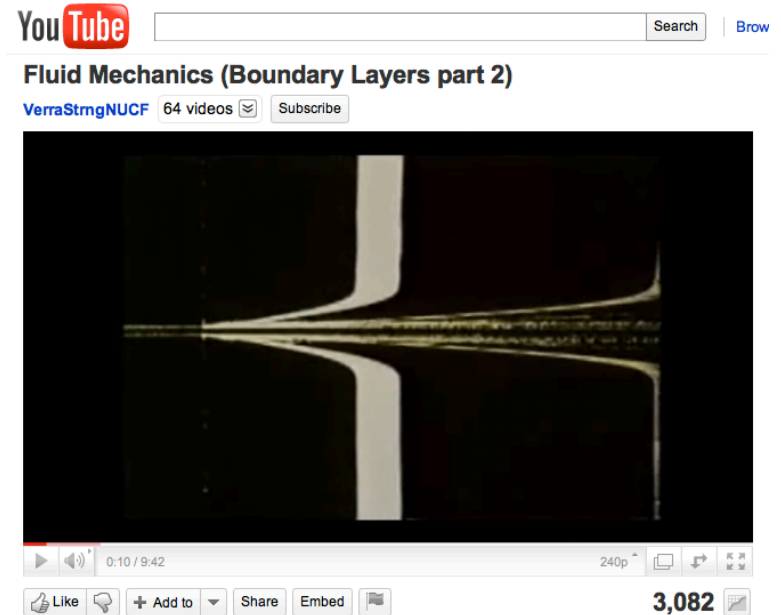
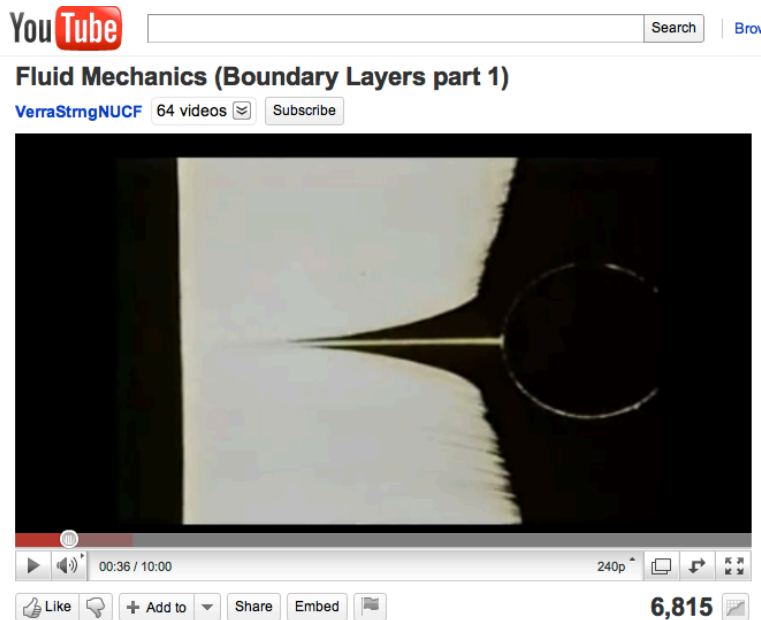
(From Harvard's Abernathy: <http://www.seas.harvard.edu/directory/fha>)
<http://www.youtube.com/watch?v=7SkVWxEUXIoM&feature=related>

Fluid Mechanics (Boundary Layers part 2)

<http://www.youtube.com/watch?v=49UsvAFKm40&feature=related>

Fluid Mechanics (Boundary Layers part 3)

<http://www.youtube.com/watch?v=WEX72jeXTGM&feature=related>



Summary

- Prandtl's thin boundary layer resolved the apparent contradiction between the usefulness of Euler's inviscid flow and the reality of the no-slip boundary condition
- For Reynolds numbers up to around 10^5 , the boundary layer is laminar
- For faster flows, or longer objects, the flow becomes turbulent.