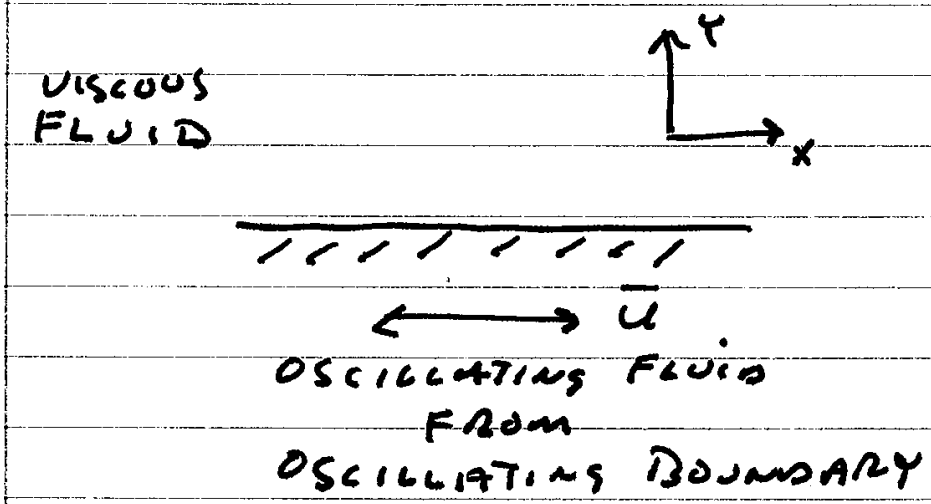
APPH 4200
Physics of Fluids

A Few More Fluid Instabilities (Ch. 12)
Turbulence (Ch. 13)
December 1, 2011

1. Viscous boundary layer and waves
2. Stability of Parallel Flows
3. Introduction to Turbulence: Lorenz Model

Viscous Boundary Layer above an Oscillating Wall

(From Landau and Lifshitz, "Fluid Mechanics", 2nd Edition, Elsevier, 2004.)



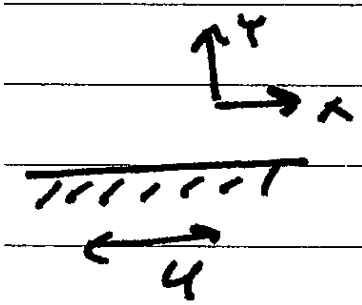
WHAT IS THE DYNAMICS OF FLUID ABOVE OSCILLATING BOUNDARY?

NAVIER-STOKES: $$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{u}$$

(WHERE HYDROSTATIC PRESSURE HAS BEEN SUBTRACTED.)

IN THE $\hat{y}$-DIRECTION, $\frac{\partial p}{\partial y} = 0$, SO PRESSURE IS CONSTAN

Equation for Viscous Flow Dynamics



X-COMPONENT OF NAVIER-STOKES:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2} \qquad \text{SINCE } (\bar{u}\cdot\bar{\nabla})\bar{u} = 0$$

BOUNDARY IS $\quad u_x(y=0) = u_0 e^{-j\omega t}$

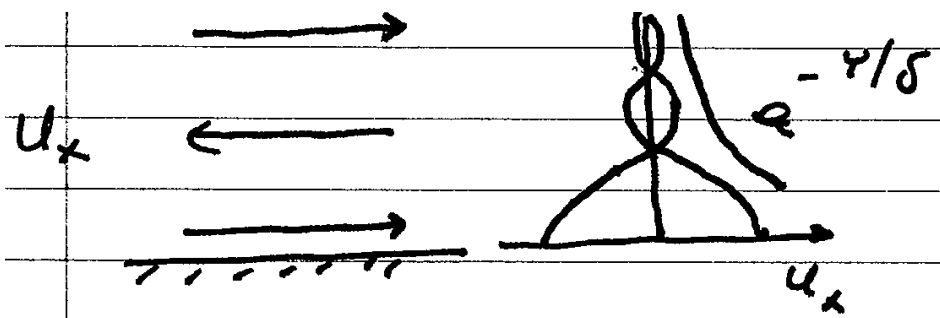
SOLUTION IS $\quad u_x(y,t) = u_0 e^{-j(\omega t - ky)}$

WITH $\quad \boxed{j\omega = \nu k^2}$

$$\sqrt{j} = \frac{1}{\sqrt{2}}(1+i) \quad \text{OR} \quad k = (1+i)\sqrt{\frac{\omega}{2\nu}} = (1+i)\,\delta^{-1}$$

$$\boxed{u_x(y,t) = u_0 \cos(\omega t - y/\delta)\, e^{-y/\delta}}$$

Penetration of Viscous Disturbances



OSCILLATIONS DAMP QUICKLY AWAY FROM BOUNDARY

THEY ALSO PROPAGATE AS "VISCOSITY" WAVES (WHICH ARE STRONGLY DAMPED)

(NOTE: "STOKES NUMBER" IS $S_t \equiv \frac{\nu}{L^2 f}$ = RATIO OF $\frac{\text{VISCOUS DAMPING RATE}}{\text{OSCILLATION FREQUENCY}}$)

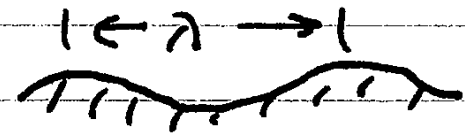
FOR THIS PROBLEM, $\delta^2 = \frac{2\nu}{\omega}$, OR $\frac{2\nu}{\omega\delta^2} \sim \frac{S_t}{\pi}$

IN ONE WAVELENGTH, $\lambda = 2\pi/k$, THE AMPLITUDE OF THE OSCILLATIONS ARE DAMPED BY A FACTOR $e^{-2\pi} = 535$!

Example: Gravity Waves

HOW BIG IS VISCOUS LAYER FOR GRAVITY WAVES?

$$\omega^2 = kg$$



$$\delta^2 = \frac{2\nu}{\omega}$$

$$k^2\delta^2 = \frac{2\nu}{\omega}k^2 = \frac{2\nu}{\sqrt{g}}k^{3/2} \quad \text{WITH } \nu \sim 10^{-6}\ m^2/s$$

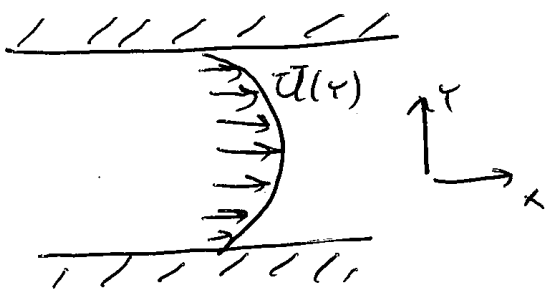
$$= 6\times10^{-7}\,k^{3/2} \sim 10^{-5}\frac{1}{\lambda^2} \quad \text{WITH } \lambda \text{ IN METERS}$$

IF $\lambda \sim 10\,m$, $\omega/2\pi = 0.4$ Hz AND $k\delta \sim 10^{-2}$!

THEREFORE, THE VISCOUS BOUNDARY LAYER IS VERY THIN!

Viscous Effects on Parallel Shear Flow

EXAMPLES:

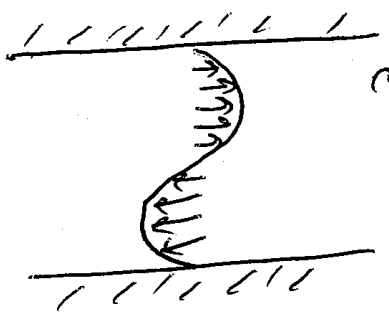


POISEUILLE FLOW

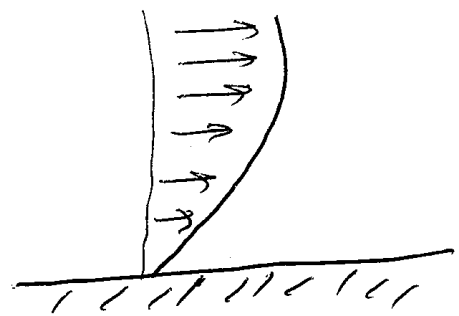
$$\bar{u} = (\bar{u}(y) + \tilde{u}_x, \tilde{u}_y)$$

$$p = \underline{p} + \tilde{p}$$

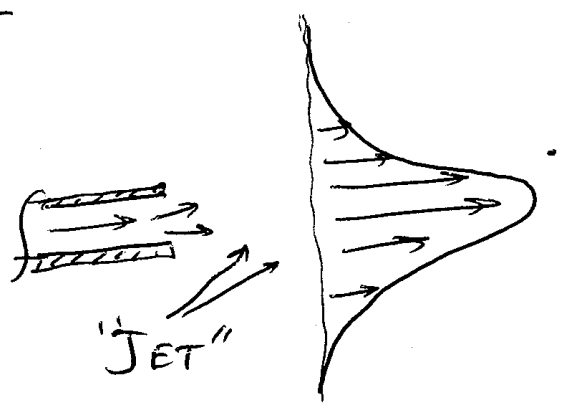
VISCOSITY MAY <u>DESTABILIZE</u> FLOW



COUNTER-STREAMING FLOW



FLOW PAST A STATIONARY WALL



"JET"

Orr-Sommerfeld Equation for 2D Dynamics in Shear Flow

EQUATIONS: $\nabla \cdot \bar{u} = 0 = \frac{\partial \tilde{u}_x}{\partial x} + \frac{\partial \tilde{u}_y}{\partial y}$ $\qquad \bar{U} = (U(y) + \tilde{u}_x, \tilde{u}_y)$

$P = \bar{P} + \tilde{p} \qquad R_e \equiv \frac{u_0 L}{\nu}$

N.S. $\hat{x}$: $$\frac{\partial \tilde{u}_x}{\partial t} + \underbrace{(U + \tilde{u}_x)\frac{\partial \tilde{u}_x}{\partial x} + \tilde{u}_y \frac{\partial}{\partial y}(U + \tilde{u}_x)}_{\text{NON-LINEAR}} = -\underbrace{\frac{\partial}{\partial x}(\bar{P} + \tilde{p})}_{\text{STATIC EQUILIBRIUM}} + \frac{1}{R_e}\underbrace{\nabla^2(U + \tilde{u}_x)}_{\text{STATIC EQUILIBRIUM}}$$

$\hat{y}$: $$\frac{\partial \tilde{u}_y}{\partial t} + \underbrace{(U + \tilde{u}_x)\frac{\partial \tilde{u}_y}{\partial x} + \tilde{u}_y \frac{\partial \tilde{u}_y}{\partial y}}_{\text{NON LINEAR}} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{R_e}\nabla^2 u_y$$

<u>STATICS</u>: $$\frac{\partial P}{\partial x} = \frac{1}{R_e}\nabla^2 U = \frac{1}{R_e}\frac{\partial^2 U}{\partial y^2}$$

<u>LINEAR DYNAMICS</u>: $$\frac{\partial \tilde{u}_x}{\partial x} + \frac{\partial \tilde{u}_y}{\partial y} = 0 \qquad \left(\nabla^2 \equiv \frac{\partial^2}{\partial y^2} - k^2\right)$$

$$\frac{\partial \tilde{u}_x}{\partial t} + U\frac{\partial \tilde{u}_x}{\partial x} + \tilde{u}_y \frac{\partial U}{\partial y} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{R_e}\nabla^2 \tilde{u}_x$$

$$\frac{\partial \tilde{u}_y}{\partial t} + U\frac{\partial \tilde{u}_y}{\partial x} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{R_e}\nabla^2 \tilde{u}_y$$

THREE UNKNOWNS: $(\tilde{u}_x, \tilde{u}_y, \tilde{p})$ AND THREE EQUATIONS.

Solution using Normal Modes

SELECT NORMAL MODES AS

$$(\tilde{u}_x, \tilde{u}_y, \tilde{p}) \sim (\text{FUNCTION OF } z) \times e^{-i(\omega t - kx)}$$

LOOK FOR $\omega = \omega_R + i\,\omega_I$ WHERE ω_R / k = PHASE VELOCITY OF DISTURBANCE

IF $\omega_I > 0$ THEN DISTURBANCE IS <u>UNSTABLE</u>

$\omega_I \leq 0$ THE STABLE OR <u>DAMPED</u>

Orr-Sommerfeld Equation

FOR INCOMPRESSIBLE 2D FLOW, USE <u>STREAMFUNCTION</u>.

$$\bar{u} = \hat{z} \times \nabla \psi \quad \text{SINCE} \quad \nabla \cdot \bar{u} = \bar{\nabla} \cdot (\hat{z} \times \nabla \psi) = 0$$

THEN $\quad \tilde{u}_x = \dfrac{\partial \psi}{\partial y} \qquad \tilde{u}_y = -\dfrac{\partial \psi}{\partial x} = -jk\psi$

[NOTICE: UNLESS $\frac{\partial \psi}{\partial y}$ AND ψ HAVE DIFFERENT COMPLEX PHASES, THEN $\tilde{u}_x(t)$ AND $\tilde{u}_y(t)$ ARE $90°$ OUT OF PHASE!]

TWO EQUATIONS AND <u>TWO</u> UNKNOWNS $(\psi, \tilde{p})$:

N.S. $\hat{x}$: $\quad -j(\omega - kU)\tilde{u}_x + \tilde{u}_y \dfrac{\partial U}{\partial y} = -jk\tilde{p} + \dfrac{1}{R_e}\nabla^2 \tilde{u}_x$

$\hat{y}$: $\quad -j(\omega - kU)\tilde{u}_y = -\dfrac{\partial \tilde{p}}{\partial y} + \dfrac{1}{R_e}\nabla^2 \tilde{u}_y \qquad \left(\nabla^2 \equiv \dfrac{\partial^2}{\partial y^2} - k^2\right)$

$\hat{x}$: $\quad (\omega - kU)\psi' + k\psi \dfrac{\partial U}{\partial y} = k\tilde{p} + \dfrac{j}{R_e}\nabla^2\left(\dfrac{\partial \psi}{\partial y}\right) \qquad \psi' \equiv \dfrac{\partial \psi}{\partial y}$

$\hat{y}$: $\quad (\omega - kU)k\psi = \dfrac{\partial \tilde{p}}{\partial y} + \dfrac{jk}{R_e}\nabla^2 \psi$

Orr-Sommerfeld Equation

ORR-SOMMERFELD EQUATION RESULTS AFTER ELIMINATING $\hat{p}$ ...

$$(\omega - kU)(k^2\psi - \psi'') - k\psi\frac{\partial^2 U}{\partial y^2} = \frac{j}{R_e}\nabla^2(k^2\psi - \psi'')$$

$$\nabla^2 = \frac{\partial^2}{\partial y^2} - k^2$$

VISCOSITY MAKES EQUATION COMPLEX AND CHANGES PHASE RELATION BETWEEN $\hat{u}_x$ AND $\tilde{u}_y$

AS $R_e \to \infty$, THE INVISCID RAYLEIGH EQUATION RESULTS

$$\boxed{(\omega - kU)(k^2\psi - \psi'') - k\psi\frac{\partial^2 U}{\partial y^2} = 0}$$

QUESTION: UNDER WHAT CONDITIONS IS $\mathcal{I}m(\omega) = \omega_I = 0$ SO THAT FLOW IS STABLE ?

Rayleigh's Inflection Point Criterion

$$\int dy\, \psi^* \left\{ (k^2\psi - \psi'') = \frac{k\psi \frac{\partial^2 U}{\partial y^2}}{\omega - kU} \right\}$$

ψ^* = COMPLEX CONJUGATE OF ψ

$$\int dy \left(k^2 |\psi|^2 - \underbrace{\frac{\partial}{\partial y}(\psi^* \psi')}_{\substack{\uparrow \\ \text{THIS TERM} \\ \text{MUST VANISH} \\ \text{AT BOUNDARIES}}} + \left|\frac{\partial \psi}{\partial y}\right|^2 \right) = \int dy\, \frac{k|\psi|^2 \frac{\partial^2 U}{\partial y^2}}{\omega - kU}$$

THEREFORE RIGHT-HAND SIDE IS REAL

<u>IMAGINARY PART OF EQUATION</u>

$$0 = \mathrm{Im}\left\{ \int dy\, (\omega^* - kU)\, k|\psi|^2 \frac{\frac{\partial^2 U}{\partial y^2}}{|\omega - kU|^2} \right\}$$

$$= \mathrm{Im}(\omega^*) \underbrace{\int dy\, k|\psi|^2 \frac{\frac{\partial^2 U}{\partial y^2}}{|\omega - kU|^2}}_{\text{MUST BE ZERO IF } \mathrm{Im}(\omega) \neq 0 \text{ !!}}$$

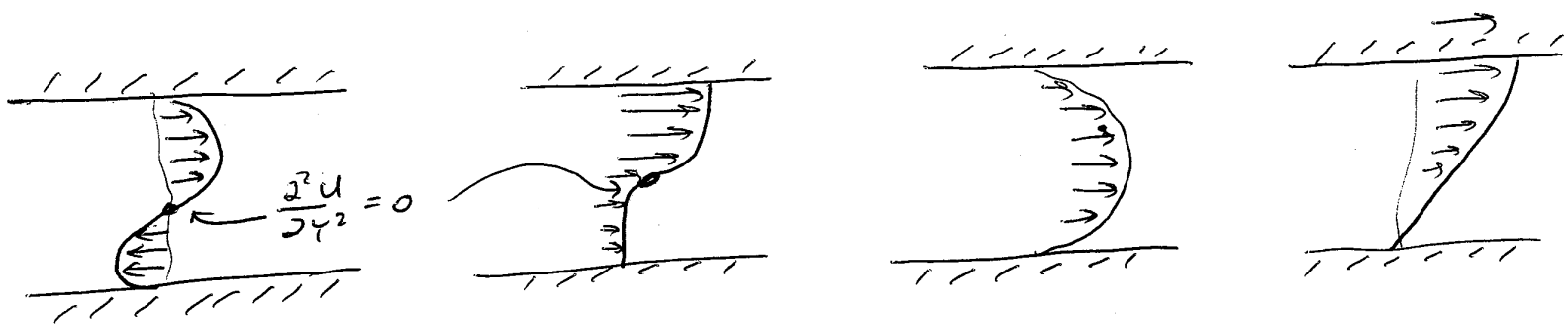
Rayleigh's Inviscid Criterion

$$\int dy \, \frac{\frac{\partial^2 U}{\partial y^2}}{|\omega - kU|^2} = 0 \quad \text{FOR SOME } (\omega, k)$$

SINCE $|\omega - kU|^2$ IS POSITIVE DEFINATE THEN

$\frac{\partial^2 U}{\partial y^2}$ MUST BE <u>BOTH</u> POSITIVE AND NEGATIVE

I.E. AN "INFLECTION" POINT



POSSIBLY UNSTABLE

POSSIBLY UNSTABLE

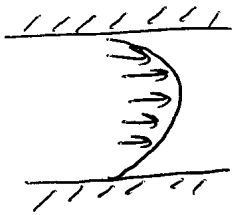
<u>STABLE</u> FOR INVISCID FLOWS!

What Happens with Finite Re?

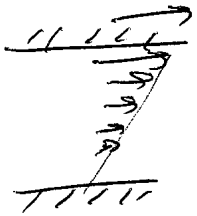
"SOLUTION OF THE ORR-SOMMERFELD EQUATION IS NOTORIOUSLY COMPLICATED" P. 498

$U(y) \sim \tanh\left(\frac{y}{L}\right)$ "SHEAR OR MIXING LAYER" UNSTABLE FOR ALL $R_o > 0$

$U(y) \sim \mathrm{sech}^2\left(\frac{y}{L}\right)$ "JET" UNSTABLE FOR $R_o > 4$



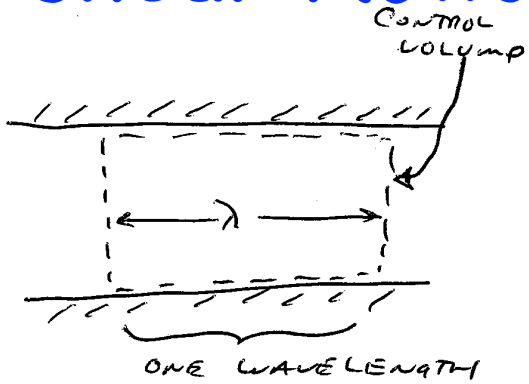
POISEUILLE FLOW UNSTABLE FOR $R_e > 5780$



COUETTE FLOW ALWAYS STABLE

Energetics of Fluctuations in Shear Flows

ENERGY (TIME-AVERAGED) EQUATION:



$$\underbrace{\iint dx\,dy\;\tilde{u}\cdot\left[\frac{\partial\tilde{u}}{\partial t}+(U+\tilde{u})\cdot\tilde{\nabla}(U+\tilde{u}) = -\frac{1}{\rho}\nabla\tilde{p}+\nu\nabla^2\tilde{u}\right]}_{+\ \text{TIME AVERAGE}}$$

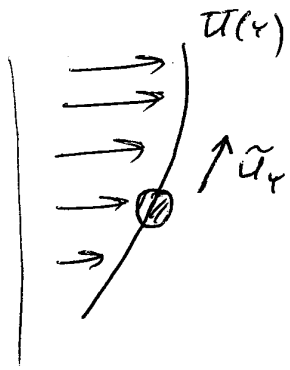
$$\frac{d}{dt}\iint dx\,dy\;\frac{1}{2}(\tilde{u})^2 = -\overline{\iint dx\,dy\;\tilde{u}\cdot\left[U\frac{\partial\tilde{u}}{\partial x}+(\tilde{u}\cdot\tilde{\nabla})\tilde{u}+(\tilde{u}\cdot\tilde{\nabla})U+\frac{1}{\rho}\tilde{u}\cdot\tilde{\nabla}\tilde{p}+\nu\,\tilde{u}\cdot\nabla^2\tilde{u}\right]}^{\text{TIME-AVERAGE}}$$

AFTER INTEGRATING BY PARTS AND USING BOUNDARY CONDITIONS THAT $\tilde{u}\to 0$ AT WALLS

TIME AVERAGE

$$\frac{d}{dt}\iint dx\,dy\;\frac{1}{2}|\tilde{u}|^2 = -\iint dx\,dy\left[\underbrace{\overline{\tilde{u}_x\tilde{u}_y}}_{\text{REYNOLDS STRESS}}\frac{\partial U}{\partial y}+\underbrace{\nu\left\{\overline{\left(\frac{\partial\tilde{u}_x}{\partial y}\right)^2}+\overline{\left(\frac{\partial\tilde{u}_y}{\partial x}\right)^2}\right\}}_{\text{VISCOUS DISSIPATION}}\right]$$

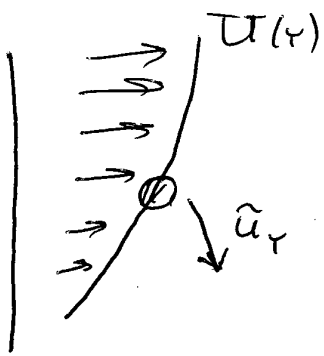
Reynolds Stress May Destabilize Shear Flow



IF $\tilde{u}_y > 0$, THEN SLOW FLUID MOVES INTO FASTER REGION.

THIS <u>SLOWS</u> AVERAGE FLOW IN X-DIRECTION

$$\therefore \quad \overline{\tilde{u}_x \tilde{u}_y} < 0$$



IF $\tilde{u}_y < 0$, THEN FAST FLUID MOVES INTO SLOWER REGION.

THIS FLOW WILL <u>INCREASE</u> AVERAGE FLOW IN X-DIRECTION.

$$\therefore \quad \overline{\tilde{u}_x \tilde{u}_y} < 0$$

$$-\iint dx\,dy\, \overline{\tilde{u}_x \tilde{u}_y} \frac{\partial U}{\partial y} > 0$$

AND THIS CAUSES FLUCTUATIONS TO <u>INCREASE</u> IN AMPLITUDE!!

Turbulence: A Grand Challenge

TURBULENCE IS A GRAND CHALLENGE IN MANY APPLICATIONS OF CONTINUUM DYNAMICS

- PLASMA PHYSICS – TRANSPORT
- ASTROPHYSICS / STELLAR CONVECTION / COLLISIONLESS SHOCKS
- GEOPHYSICS / WEATHER / CIRCULATION / MIXING WAVES
- FLUID DYNAMICS / DRAG / NOZZELS AND JETS

TURBULENCE THEORY IS VERY DIFFICULT

- NONLINEAR
- STRONGLY MODIFIES INITIAL CONDITIONS
- COUPLES ENERGY FROM ONE SCALE TO ANOTHER

TURBULENCE MODELS USE A STATISTICAL APPROACH

- RANDOM / CHAOTIC
- HOW DO AVERAGES AND CORRELATIONS EVOLVE?

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

1. Introduction

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and, even when observed for long periods of time, do not appear to repeat their previous history.

5. The convection equations of Saltzman

In this section we shall introduce a system of three ordinary differential equations whose solutions afford the simplest example of deterministic nonperiodic flow of which the writer is aware. The system is a simplification of one derived by Saltzman (1962) to study finite-amplitude convection. Although our present interest is in the nonperiodic nature of its solutions, rather than in its contributions to the convection problem, we shall describe its physical background briefly.

Rayleigh (1916) studied the flow occurring in a layer of fluid of uniform depth H, when the temperature difference between the upper and lower surfaces is maintained at a constant value ΔT. Such a system possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth, If this solution is unstable, convection should develop.

In the case where all motions are parallel to the $x-z$-plane, and no variations in the direction of the y-axis occur, the governing equations may be written (see Saltzman, 1962)

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}, \tag{17}$$

$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta. \tag{18}$$

Here ψ is a stream function for the two-dimensional motion, θ is the departure of temperature from that occurring in the state of no convection, and the constants g, α, ν, and κ denote, respectively, the acceleration of gravity, the coefficient of thermal expansion, the kinematic viscosity, and the thermal conductivity. The problem is most tractable when both the upper and lower boundaries are taken to be free, in which case ψ and $\nabla^2\psi$ vanish at both boundaries.

Rayleigh found that fields of motion of the form

$$\psi = \psi_0 \sin(\pi a H^{-1} x) \sin(\pi H^{-1} z), \tag{19}$$

$$\theta = \theta_0 \cos(\pi a H^{-1} x) \sin(\pi H^{-1} z), \tag{20}$$

would develop if the quantity

$$R_a = g\alpha H^3 \Delta T \nu^{-1} \kappa^{-1}, \tag{21}$$

now called the *Rayleigh number*, exceeded a critical value

$$R_c = \pi^4 a^{-2} (1 + a^2)^3. \tag{22}$$

The minimum value of R_c, namely $27\pi^4/4$, occurs when $a^2 = \frac{1}{2}$.

Saltzman (1962) derived a set of ordinary differential equations by expanding ψ and θ in double Fourier series in x and z, with functions of t alone for coefficients, and substituting these series into (17) and (18). He arranged the right-hand sides of the resulting equations in double-Fourier-series form, by replacing products of trigonometric functions of x (or z) by sums of trigonometric functions, and then equated coefficients of similar functions of x and z. He then reduced the resulting infinite system to a finite system by omitting reference to all but a specified finite set of functions of t, in the manner proposed by the writer (1960).

These same solutions would have been obtained if the series had at the start been truncated to include a total of three terms. Accordingly, in this study we shall let

$$a(1+a^2)^{-1}\kappa^{-1}\psi = X\sqrt{2}\sin(\pi a H^{-1}x)\sin(\pi H^{-1}z), \tag{23}$$

$$\pi R_c^{-1}R_a\Delta T^{-1}\theta = Y\sqrt{2}\cos(\pi a H^{-1}x)\sin(\pi H^{-1}z) - Z\sin(2\pi H^{-1}z), \tag{24}$$

where X, Y, and Z are functions of time alone. When expressions (23) and (24) are substituted into (17) and (18), and trigonometric terms other than those occurring in (23) and (24) are omitted, we obtain the equations

$$X^{\cdot} = -\sigma X + \sigma Y, \tag{25}$$

$$Y^{\cdot} = -XZ + rX - Y, \tag{26}$$

$$Z^{\cdot} = XY - bZ. \tag{27}$$

Here a dot denotes a derivative with respect to the dimensionless time $\tau = \pi^2 H^{-2}(1+a^2)\kappa t$, while $\sigma = \kappa^{-1}\nu$ is the *Prandtl number*, $r = R_c^{-1}R_a$, and $b = 4(1+a^2)^{-1}$. Except for multiplicative constants, our variables X, Y, and Z are the same as Saltzman's variables A, D, and G. Equations (25), (26), and (27) are the convection equations whose solutions we shall study.

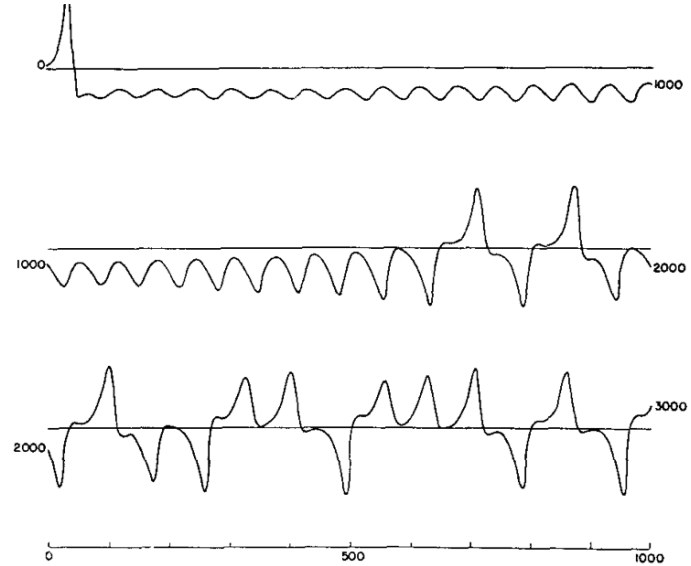
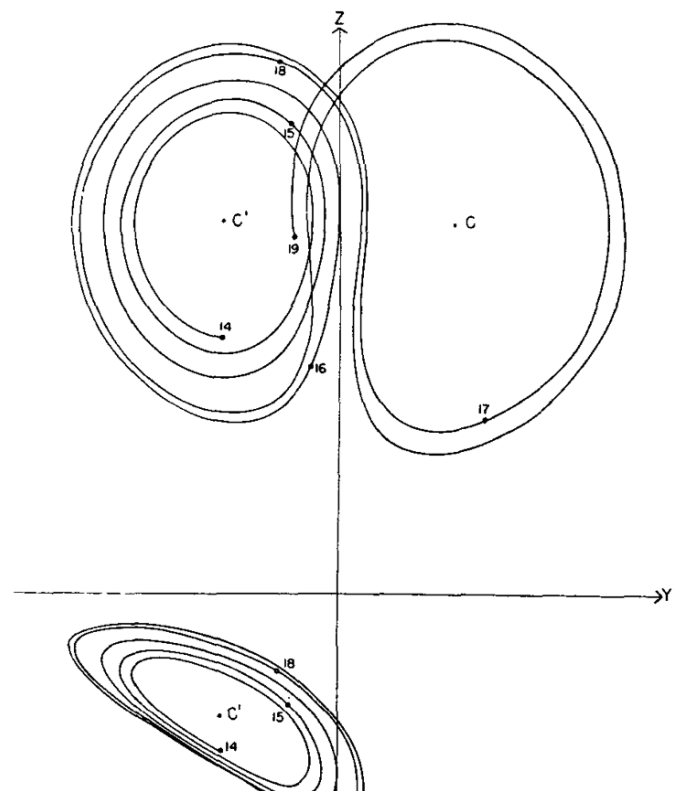


FIG. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

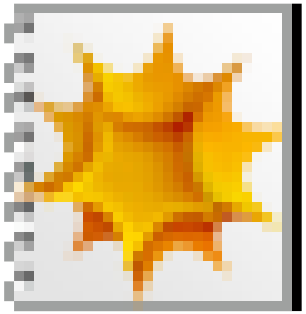


8. Conclusion

Certain mechanically or thermally forced nonconservative hydrodynamical systems may exhibit either periodic or irregular behavior when there is no obviously related periodicity or irregularity in the forcing process. Both periodic and nonperiodic flow are observed in some experimental models when the forcing process is held constant, within the limits of experimental control. Some finite systems of ordinary differential equations designed to represent these hydrodynamical systems possess periodic analytic solutions when the forcing is strictly constant. Other such systems have yielded nonperiodic numerical solutions.

When our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent.

Mathematica Notebook



Lorenz_Model.nb

Summary

- Understanding linear instabilities in fluid dynamics involves three important steps:
  - Continuity and Navier-Stokes
  - Statics
  - Linear Dynamics
  - Reduction using normal modes
  - Matching boundary conditions
- Flow shear instabilities can be destabilized by viscosity!
- Nonlinearity, $U \cdot \nabla U$, can drive chaotic, or turbulent, dynamics