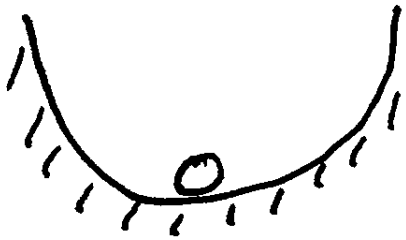
APPH 4200
Physics of Fluids

Fluid Instabilities (Ch. 12)

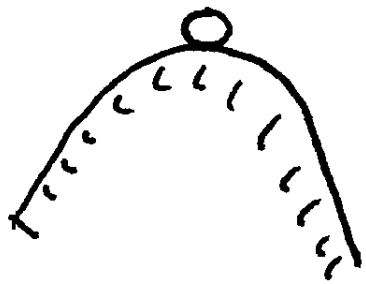
1. Introduction
2. Bénard Thermal Instability

Equilibria can be Stable or Unstable

FOUR TYPES OF EQUILIBRIA...



STABLE



UNSTABLE

NEUTRAL

NON-LINEAR
UNSTABLE

Linear Instability

METHOD OF NORMAL MODES

$$\{\rho, u, P, \ldots\} \propto e^{st}\, \underbrace{g(x, y, z)}$$

IF $s > 0$, THEN INSTABILITY

MUST SATISFY BOUNDARY CONDITIONS

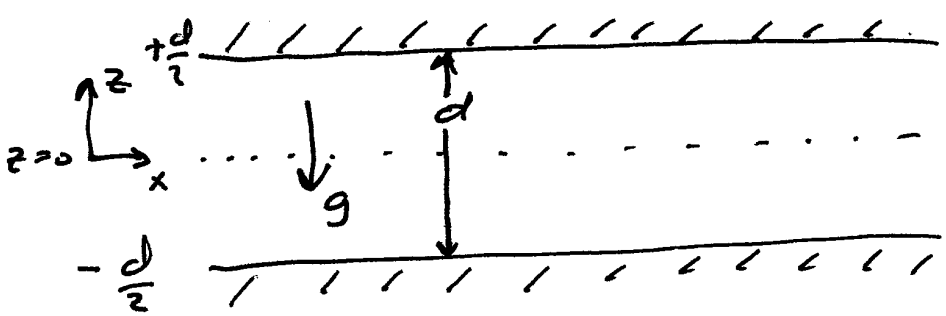
IN MANY CASES, $g(x, y, z)$ CAN BE REPRESENTED BY "PLANE-WAVE" OSCILLATIONS: $e^{j(k_x x + k_z y)}$

THE COMPLICATED PDE'S FOR FLUID DYNAMICS CAN BE TURNED INTO AN ALGEBRAIC OR ODE EQUATION FOR s IN TERMS OF SPATIAL VARIATION OF MODES, $g(x, y, z)$

Henri Bénard, (1874-1939)

Henri Bénard was a French physicist who performed experiments on fluids for a Collège de France physics course given by Marcel Brillouin at the turn of the century. Bénard was among the first to study the behavior of a thin layer of liquid, about a millimeter in depth, when heated from below, the upper surface being in contact with air at a lower temperature. Experimenting with liquids of different viscosity, he observed in all cases the formation of a striking pattern of hexagonal cells. **In his 1900 article, Bénard used a variety of means to visualize the structures he wanted to exhibit. They ranged from material substances he added to the liquid to optical contrivances such as lighting and the design of special photographic setups. His papers were abundantly illustrated with sketches and photographic clichés.** In 1916, Lord Rayleigh provided a mathematical explanation for the onset of instability in such a convective system.

Bénard Thermal Instability



"COLD" PLATE $T_0 - \Delta T$

DEFINE $\Gamma \equiv \frac{\Delta T}{d}$

AVERAGE TEMPERATURE GRADIENT

"HOT" PLATE T_0

USE "BOUSSINESQ" APPROXIMATION FOR FLUID DYNAMICS

ASSUME $\nabla \cdot \bar{U} = 0$

AND $\rho(T) = \rho_0 \left[1 - \alpha (T - T_0) \right]$

$\alpha > 0$ MEANS HOT FLUID IS LESS DENSE THAN COLD FLUID.

$\therefore$ HOT FLUID RISES FROM BOTTOM PLATE

COLD FLUID FALLS FROM UPPER PLATE

THERMAL DIFFUSIVITY AND VISCOSITY DAMP CONVECTIVE FLUID MOTION

Equations of Fluid Dynamics

NAVIER-STOKES: $$\rho_0 \frac{D\bar{u}}{Dt} = -\bar{\nabla} p - g\rho(T)\hat{z} + \mu \nabla^2 \bar{u}$$

THERMAL DIFFUSION: $$\frac{DT}{Dt} = \kappa \nabla^2 T$$

INCOMPRESSIBLE: $$\nabla \cdot \bar{u} = 0$$

<u>DYNAMICAL VARIABLES</u>:

$$\bar{u}(x, y, z, t)$$

$$T = \bar{T}(z) + \tilde{T}(x, y, z, t)$$

$$p = \bar{p}(z) + \tilde{p}(x, y, z, t)$$

$\tilde{T}, \tilde{p}$ ARE FLUCTUATING VARIABLES

$\bar{T}, \bar{p}$ ARE <u>STATIC</u> FUNCTIONS OF HYDROSTATIC EQUILIBRIUM

<u>SOLUTION</u>:

- FIND HYDROSTATIC RELATIONSHIP
- SOLVE <u>LINEARIZED</u> EQUATIONS FOR EIGENVALUE, s, AND FOR "NORMAL MODES" THAT SATISFY BOUNDARY CONDITIONS

Statics

FORCE BALANCE: $$0 = -\frac{\partial \bar{P}}{\partial z} - \rho_0 g \left(1 - \alpha\left(\bar{T}(z) - T_0\right)\right)$$

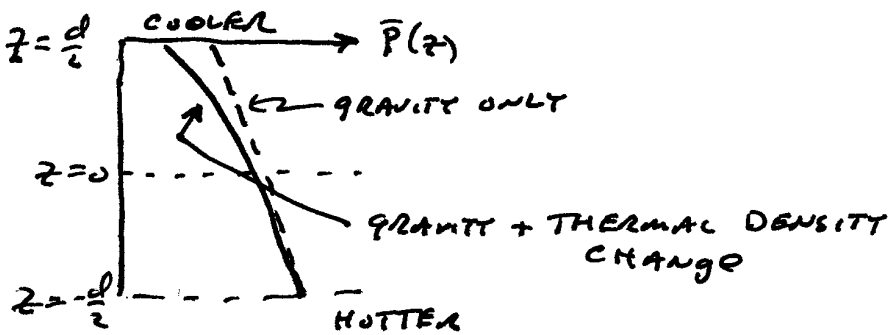
THERMAL FLUX: $$0 = \kappa \frac{\partial^2 \bar{T}}{\partial z^2}$$ (CONSTANT HEAT FLUX)

STATIC TEMPERATURE PROFILE:

$$\bar{T}(z) = T_0 - \left(\frac{\Delta T}{d}\right)\left(z + \frac{d}{2}\right) \qquad \begin{cases} \bar{T}(z = -\frac{d}{2}) = T_0 \\ \bar{T}(z = +\frac{d}{2}) = T_0 - \Delta T \end{cases}$$

CONDITION FOR STATIC FORCE BALANCE:

$$\frac{\partial \bar{P}}{\partial z} = -\rho_0 g \left[1 - \alpha\left(T_0 - \frac{\Delta T}{d}\left(z + \frac{d}{2}\right) - T_0\right)\right]$$
$$= -\rho_0 g \left[1 + \alpha \frac{\Delta T}{d}\left(z + \frac{d}{2}\right)\right]$$



Nonlinear Equations for Perturbations

SUBTRACT HYDROSTATIC CONDITION FROM EQUATIONS FOR FLUID DYNAMICS TO GET NON-LINEAR EQUATIONS FOR PERTURBATIONS.

$$\frac{\partial u_z}{\partial t} + (\bar{u}\cdot\bar{\nabla})\, u_z = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial z} + g\alpha\tilde{T} + \nu\nabla^2 u_z$$

$$\frac{\partial u_x}{\partial t} + (\bar{u}\cdot\bar{\nabla})\, u_x = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} + \nu\nabla^2 u_x$$

$$\frac{\partial u_y}{\partial t} + (\bar{u}\cdot\bar{\nabla})\, u_y = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y} + \nu\nabla^2 u_y$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial \tilde{T}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\left(\bar{T}(z) + \tilde{T}\right) = \kappa\nabla^2\tilde{T}$$

UNKNOWN FUNCTIONS: $u_x,\ u_y,\ u_z,\ \tilde{p},\ \tilde{T}$

5 EQUATIONS + 5 UNKNOWNS

Linear Equations

(A) $$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial z} + g\alpha\tilde{T} + \nu\nabla^2 u_z$$

(B) $$\frac{\partial u_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} + \nu\nabla^2 u_x$$

(C) $$\frac{\partial u_y}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y} + \nu\nabla^2 u_y$$

(D) $$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

(E) $$\frac{\partial \tilde{T}}{\partial t} + u_z \underbrace{\frac{\partial \bar{T}}{\partial z}}_{\Gamma = -\frac{\Delta T}{d}} = \kappa\nabla^2\tilde{T}$$

PLAN FOR SIMPLIFICATION:

- RELATE $\tilde{p}$ AND $\tilde{T}$ BY COMBINING EQS A, B, C, D
- ELIMINATE $\tilde{p}$ FROM A BY USING RELATIONSHIP ABOVE
- SOLVE TWO SIMULTANEOUS PDES FOR $\tilde{u}_z$ AND $\tilde{T}$

Simplification

- COMBINE A, B, C, D ...

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)}_{\nabla\cdot\bar{u}} = -\frac{1}{\rho_0}\left(\frac{\partial^2\tilde{p}}{\partial x^2}+\frac{\partial^2\tilde{p}}{\partial y^2}+\frac{\partial^2\tilde{p}}{\partial z^2}\right) + g\alpha\frac{\partial\tilde{T}}{\partial z} + \nu\nabla^2(\nabla\cdot\bar{u})$$

DOT, $\nabla\cdot\bar{u}=0$, SO

$$\boxed{\frac{1}{\rho_0}\nabla^2\tilde{p} = g\alpha\frac{\partial\tilde{T}}{\partial z}}$$

- TAKE LAPLACIAN OF A ....

$$\frac{\partial}{\partial t}\nabla^2 u_z = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\nabla^2\tilde{p} + g\alpha\nabla^2\tilde{T} + \nu\nabla^2\nabla^2 u_z$$

- ELIMINATE $\tilde{p}$ ...

$$\boxed{\frac{\partial}{\partial t}\nabla^2 u_z = g\alpha\left(\nabla^2\tilde{T}-\frac{\partial^2\tilde{T}}{\partial z^2}\right) + \nu\nabla^2\nabla^2 u_z}$$

Two Coupled PDEs forms a Linear Eigensytem

PERTURBED VERTICAL VELOCITY:

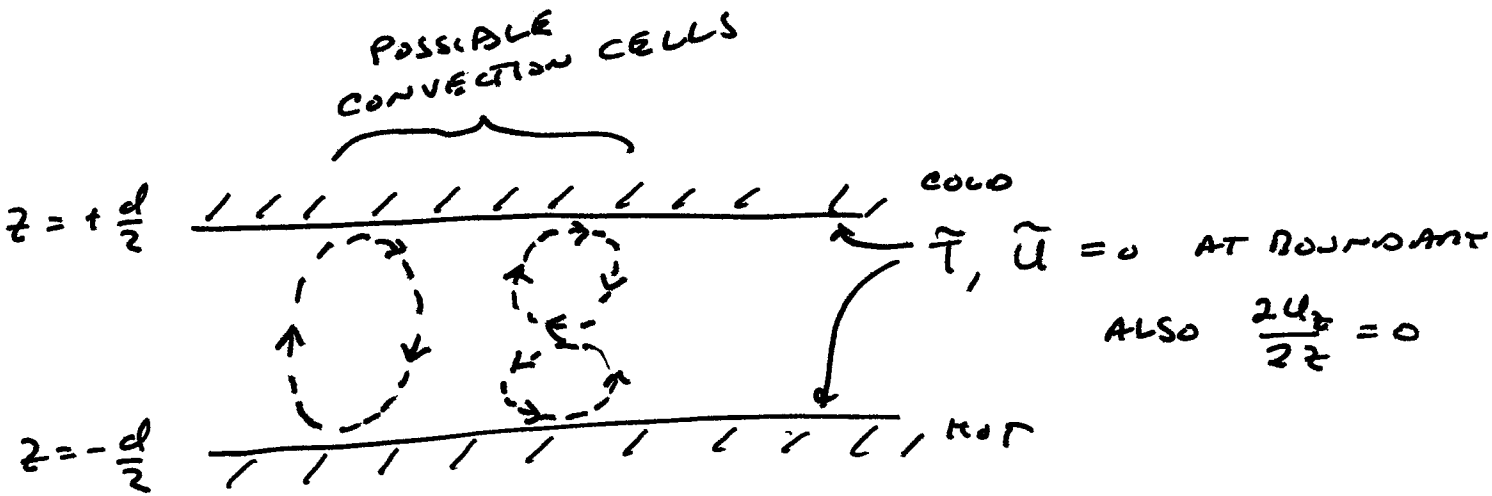
$$\frac{\partial}{\partial t}\nabla^2 u_z - \nu \nabla^2 \nabla^2 u_z = g\alpha\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\tilde{T}$$

PERTURBED THERMAL DIFFUSION:

$$\frac{\partial}{\partial t}\tilde{T} - \kappa \nabla^2 \tilde{T} = +u_z \Gamma \qquad \left(\Gamma \equiv \frac{dT}{dz}\right)$$

ONCE $u_z(x, y, z, t)$ AND $\tilde{T}(x, y, z, t)$ ARE KNOWN, WE CAN FIND THE FORMS FOR $(\tilde{p}, u_x, u_y)$.

Boundary Conditions



- $\tilde{T} = 0$ At $z = \pm \frac{d}{2}$ (T SET BY PLATES, CONSTANT)
- $(\tilde{u}_x, u_y, u_z) = 0$ AT $z = \pm \frac{d}{2}$ (NO SLIP BOUNDARIES)
- SINCE $(u_x, u_y) = 0$ EVERYWHERE AT BOUNDARY...

  THEN $\nabla \cdot u = 0$ IMPLIES $\frac{\partial u_z}{\partial z} = -\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial z}\right) = 0$

Normal Modes: Rewrite PDE as ODE

LET $u_z(x, y, z, t) = \hat{u}_z(z)\, e^{st}\, e^{j(k_x x + k_y y)}$

$$\hat{T}(x, y, z, t) = \hat{T}(z)\, e^{st}\, e^{j(k_x x + k_y y)}$$

$$k^2 \equiv k_x^2 + k_y^2$$

NOTE: IF $Re(s) > 0$, <u>UNSTABLE</u> growing mode

IF $Re(s) < 0$, DAMPED, <u>STABLE</u> MODE

IF $Im(s) \neq 0$, MODES ARE OSCILLATORY, "WAVE-LIKE"

NOTE:

$$\frac{\partial}{\partial t} \rightarrow s$$

$$\nabla^2 \rightarrow \frac{\partial^2}{\partial z^2} - k^2$$

PDE$(x, y, z, t) \rightarrow$ ODE(z)

Mode Equations

PERTURBED VERTICAL VELOCITY :

$$s\left(\frac{\partial^2 \hat{u}_z}{\partial z^2} - k^2 \hat{u}_z\right) - \nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 \hat{u}_z = -g\alpha k^2 \tilde{T}$$

PERTURBED THERMAL DIFFUSION :

$$s\,\hat{T} - \kappa\left(\frac{\partial^2 \tilde{T}}{\partial z^2} - k^2 \hat{T}\right) = +\hat{u}_z \Gamma'$$

A MORE CONVENIENT FORM:

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 \hat{u}_z - \frac{s}{\nu}\left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{u}_z = \frac{g\alpha}{\nu} k^2 \hat{T}$$

$$\frac{\partial^2 \hat{T}}{\partial z^2} - \left(k^2 + \frac{s}{\kappa}\right)\hat{T} = -\frac{\Gamma}{\kappa}\hat{u}_z$$

NEED TO:

- FIND FORM OF $u_z(z)$ THAT SATISFIES BOUNDARY (i.E. MODES)
- FIND "EIGENVALUES" OR RELATIONSHIP BETWEEN s AND k^2 AND MODE STRUCTURE

Dimensionless Form

LENGTH $\sim d$

TIME $\sim d^2/\kappa$ (THE THERMAL DIFFUSION TIME SCALE)

TEMPERATURE $\sim \Delta T$

VELOCITY $\sim d/(d^2/\kappa) \sim (\kappa/d)$

THEN, $$\frac{\partial^2 \hat{T}}{\partial z^2} \longrightarrow \frac{\Delta T}{d^2} \underbrace{\frac{\partial^2 \hat{T}}{\partial z^2}}_{\text{DIMENSIONLESS FORM}}$$

$$\left(k^2 + \frac{s}{\kappa}\right)\hat{T} \longrightarrow \frac{\Delta T}{d^2} \underbrace{\left(k^2 + s\right)\hat{T}}_{\text{DIMENSIONLESS FORM}}$$

$$u_z \frac{\Gamma}{\kappa} \longrightarrow \frac{\Gamma}{d}\left(\frac{\kappa}{d}\right) \underbrace{\hat{u}_z}_{\text{DIMENSIONLESS}} = \frac{\Delta T}{d^2}\hat{u}_z$$

ETC ---.

Dimensionless Equations

$$\left[\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 - \frac{s}{P_R}\left(\frac{\partial^2}{\partial z^2} - k^2\right)\right]\hat{u}_z = R_a k^2 \hat{T}$$

$$\frac{\partial^2 \hat{T}}{\partial z^2} - (k^2 + s)\hat{T} = -\hat{u}_z$$

WHERE $P_R \equiv \frac{\nu}{\kappa}$ = PRANDTL NUMBER = $\frac{\text{MOMENTUM DIFFUSION (VISCOSITY)}}{\text{HEAT DIFFUSION}}$

$R_a = \frac{g \alpha \Delta T d^3}{\nu \kappa}$ = RAYLEIGH NUMBER = $\frac{\text{BUOYANCY FORCE}}{\text{DIFFUSION FORCE}}$

ONLY <u>TWO</u> NUMBERS PARAMETERIZE DYNAMICAL BEHAVIOR!

BOUNDARIES:

$$\hat{T}\left(z = \pm\tfrac{1}{2}\right) = 0$$

$$\hat{u}_z\left(z = \pm\tfrac{1}{2}\right) = 0$$

$$\frac{\partial \hat{u}_z}{\partial z}\left(z = \pm\tfrac{1}{2}\right) = 0$$

Marginal Instability/Stability

LET $S=0$, FIND RELATIONSHIP BETWEEN MODES AND PARAMETERS (R_a, P_r) FOR THRESHOLD FOR INSTABILITY.

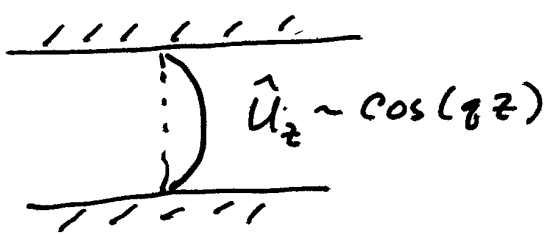
IF $S=0$, THEN

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 \hat{u}_z = R_a k^2 \hat{T}$$

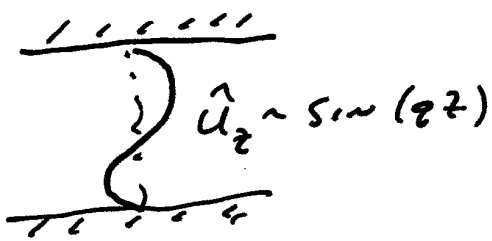
$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) \hat{T} = -\hat{u}_z$$

OR

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^3 \hat{u}_z = -R_a k^2 \hat{u}_z$$



EVEN MODE



ODD MODE

NOTE:

- q IS COMPLEX
- 6 ROOTS FOR q BECAUSE EQ FOR u_z IS 6TH-ORDER ODE!

Solution for Marginal Instability/Stability

EVEN MODE

$$\hat{u}_z \sim \cos(qz)$$

$$\frac{\partial^2 u_z}{\partial z^2} = -q^2 \cos(qz)$$

ODD MODE

$$\hat{u}_z \sim \sin(qz)$$

$$\frac{\partial^2 u_z}{\partial z^2} \sim -q^2 \sin(qz)$$

YIELDS GREAT SIMPLIFICATION . . .

$$\boxed{\left(q^2 + k^2\right)^3 = R_a k^2}$$

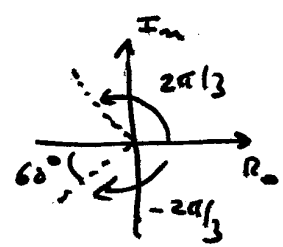
$$R_a \equiv \frac{g \alpha \Delta T d^3}{\nu \kappa}$$

SIX ROOTS FOR q . . .

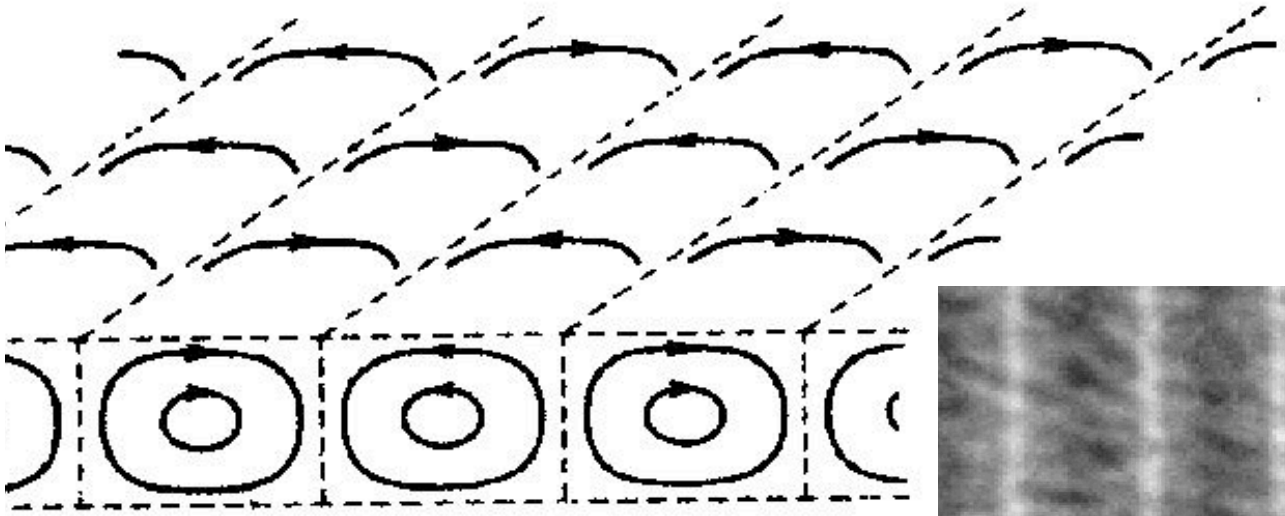
$$q^2 = -k^2 + (1)^{1/3} \sqrt[3]{R_a k^2}$$

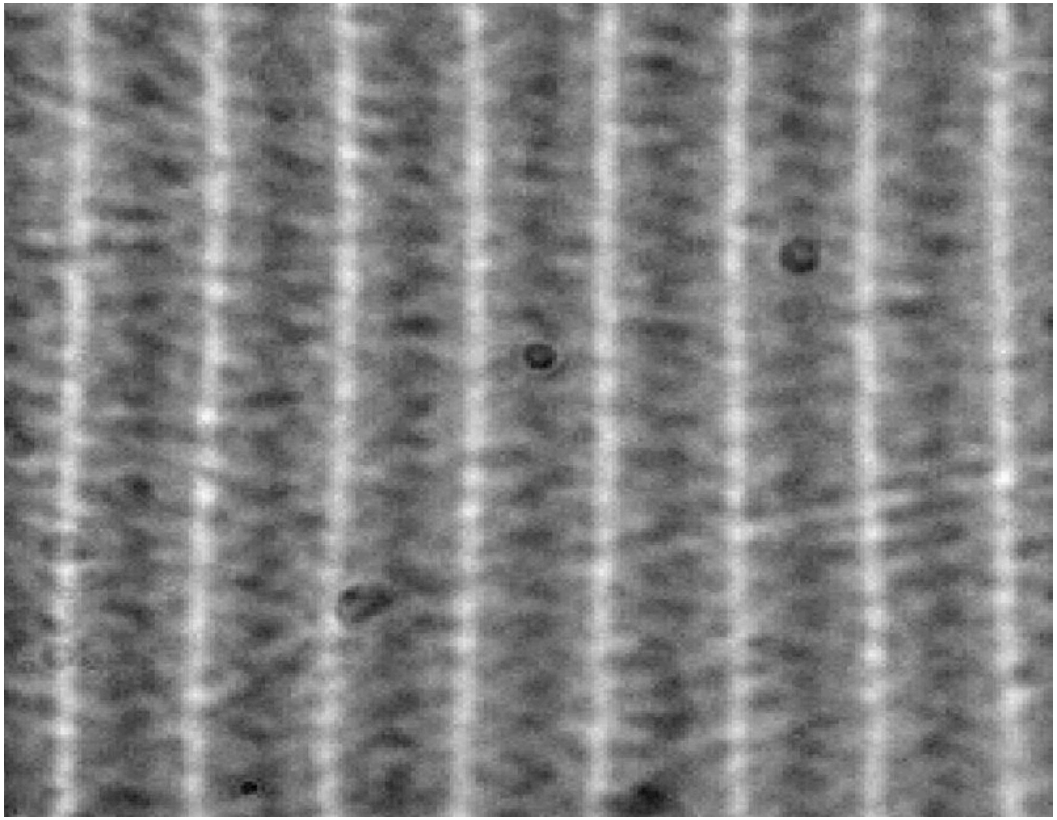
WHERE

$$(1)^{1/3} = \begin{cases} 1^{1/3} = 1 \\ e^{i 2\pi/3} = -\frac{1}{2}\left(1 \pm \sqrt{3}\, i\right) \\ e^{-i 2\pi/3} = -\frac{1}{2}\left(1 - \sqrt{3}\, i\right) \end{cases}$$

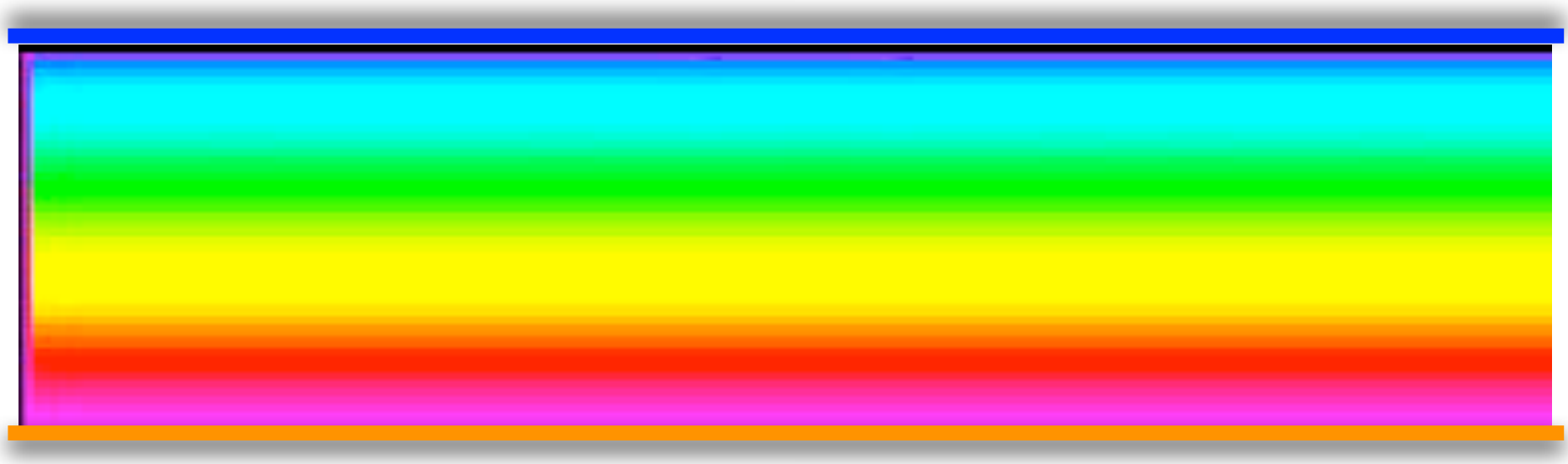


Bénard Photos





Fun 2D Simulation Shows Linear to Nonlinear Evolution



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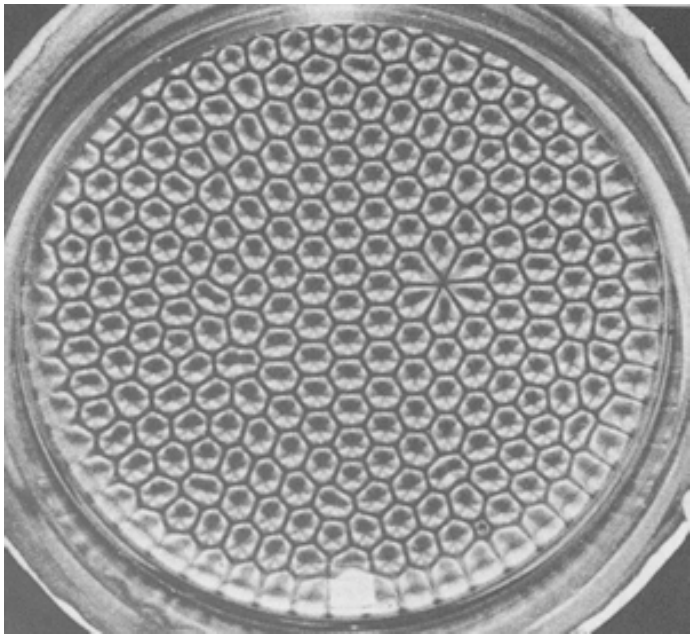
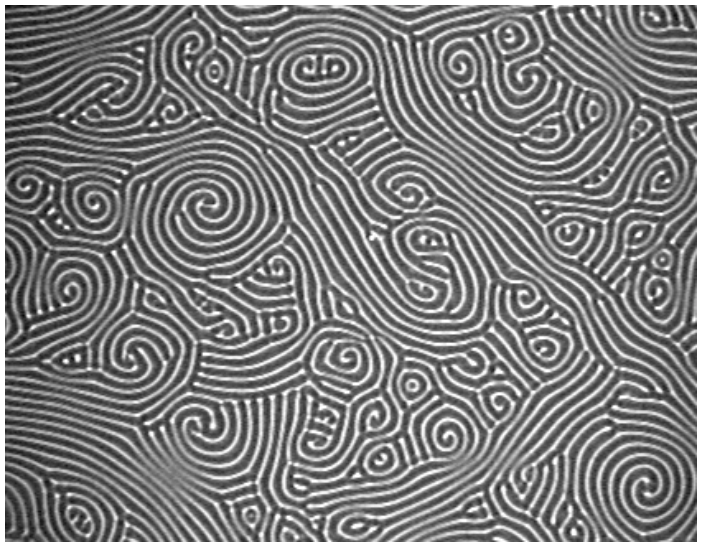
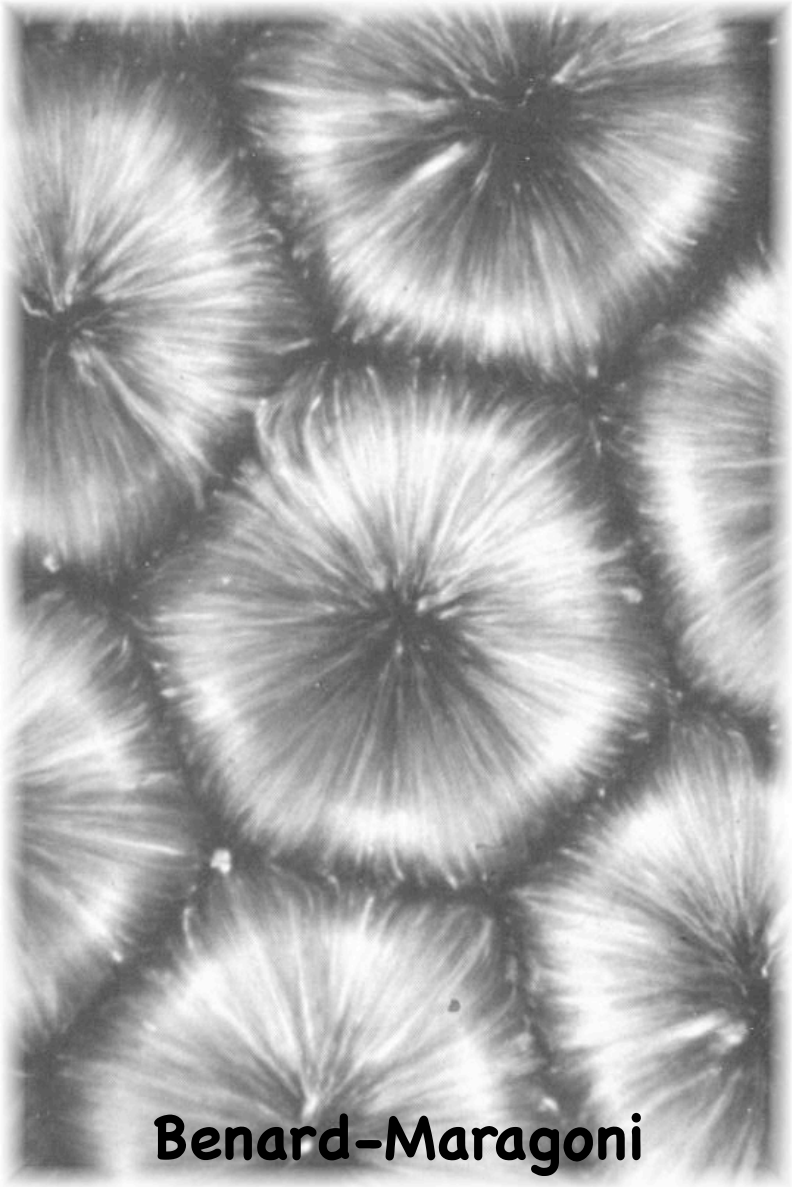
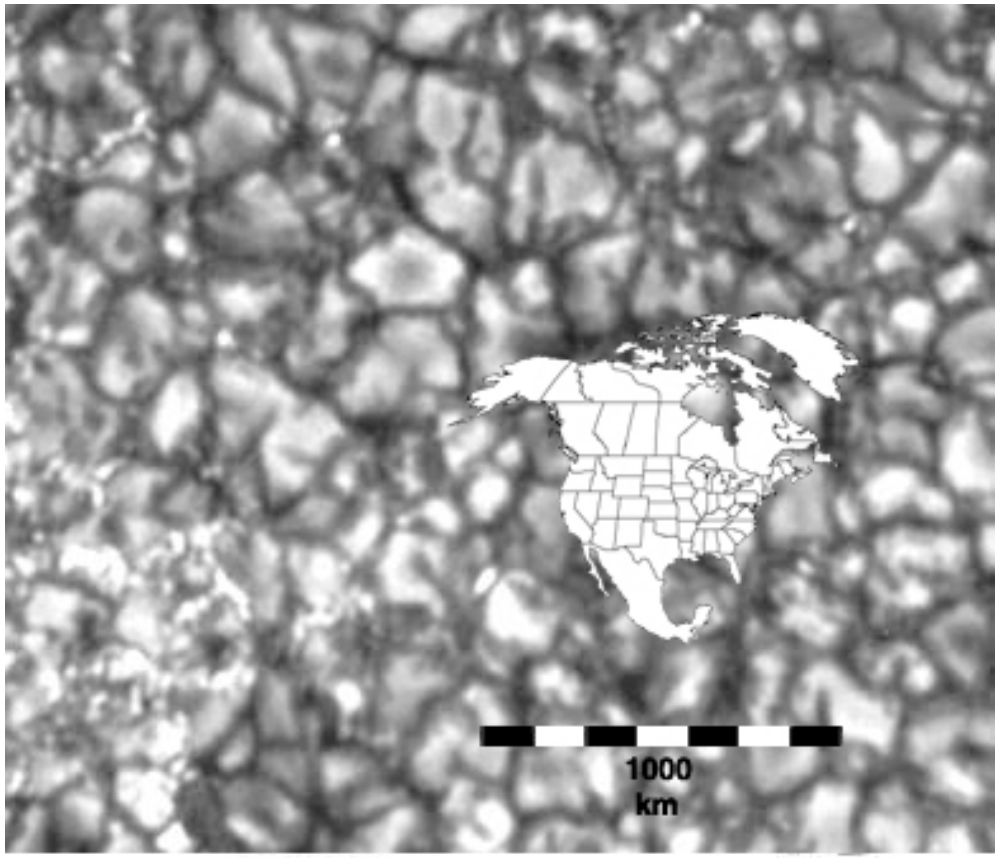






Solar Granulation



Photospheric granulation, G. Scharmer
Swedish Vacuum Solar Telescope
10 July 1997

Mathematica Notebook

Bernard-Marginal-Mode.nb

Summary

- Instabilities are (initially) small perturbations that grow in time.
- The method used to find the mode structure and growth time for a linear instability is similar to the method used to find wave structure and dispersion.
- Bénard-Rayleigh thermal instability occurs when the temperature gradient exceeds a threshold.
- As the ΔT increases, a single mode becomes unstable first.