1. Introduction
2. Bénard Thermal Instability
Equilibria can be Stable or Unstable

Four types of equilibria...

- Stable
- Unstable
- Neutral
- Non-linear unstable
Linear Instability

METHOD OF NORMAL MODES

\[
\begin{bmatrix}
P, U, P, \ldots \end{bmatrix} \propto e^{st} g(x, y, z)
\]

IF \( s > 0 \), THEN

INSTABILITY

\[
g(x, y, z) \text{ must satisfy boundary conditions}
\]

In many cases, \( g(x, y, z) \) can be represented by

"plane-wave" oscillations:

\[
\psi(x, y) = e^{i(k_x x + k_y y)}
\]

The complicated PDE's for fluid dynamics can be turned into an algebraic or ODE equation for \( s \) in terms of spatial variation of modes, \( g(x, y, z) \)
Henri Bénard, (1874–1939)

Henri Bénard was a French physicist who performed experiments on fluids for a Collège de France physics course given by Marcel Brillouin at the turn of the century. Bénard was among the first to study the behavior of a thin layer of liquid, about a millimeter in depth, when heated from below, the upper surface being in contact with air at a lower temperature. Experimenting with liquids of different viscosity, he observed in all cases the formation of a striking pattern of hexagonal cells. In his 1900 article, Bénard used a variety of means to visualize the structures he wanted to exhibit. They ranged from material substances he added to the liquid to optical contrivances such as lighting and the design of special photographic setups. His papers were abundantly illustrated with sketches and photographic clichés. In 1916, Lord Rayleigh provided a mathematical explanation for the onset of instability in such a convective system.
Bénard Thermal Instability

Use "Boussinesq" Approximation for Fluid Dynamics

Assume $\nabla \cdot \bar{U} = 0$

And

$\rho(T) = \rho_0 \left[ 1 - \alpha(T - T_0) \right]$

$\alpha > 0$ means hot fluid is less dense than cold fluid.

Hence hot fluid rises from bottom plate

Cold fluid falls from upper plate

Thermal diffusivity and viscosity damp convection fluid motion
Equations of Fluid Dynamics

Navier-Stokes: \( \rho_0 \frac{D\mathbf{U}}{Dt} = -\nabla \rho - g \rho(t) \mathbf{e} + \mu \nabla^2 \mathbf{U} \)

Thermal Diffusion: \( \frac{DT}{Dt} = k \nabla^2 T \)

Incompressible: \( \nabla \cdot \mathbf{U} = 0 \)

Dynamical Variables:

\( \bar{U}(x, y, z, t) \)

\( T = \bar{T}(z) + \tilde{T}(x, y, z, t) \) \( \bar{T}, \tilde{T} \) are fluctuating variables

\( P = \bar{P}(z) + \tilde{P}(x, y, z, t) \) \( \bar{P}, \tilde{P} \) are static variables

Solution:

- Find Hydrostatic Relationship
- Solve Linearized Equations for Eigenvalue, \( \lambda \), and for "normal modes" that satisfy boundary conditions
Statics

Force Balance:
\[ O = -\frac{2\bar{P}}{2\varepsilon} - p_0 g \left( 1 - \alpha \left( \bar{T}(z) - T_0 \right) \right) \]

Thermal Flux:
\[ O = \lambda \frac{2 \bar{T}}{2 \varepsilon^2} \quad \text{(constant heat flux)} \]

Static Temperature Profile:
\[ \bar{T}(z) = T_0 - \left( \frac{\Delta T}{d} \right) \left( z + \frac{d}{2} \right) \quad \begin{cases} \bar{T}(z = -\frac{d}{2}) = T_0 \\ \bar{T}(z = \frac{d}{2}) = T_0 - \Delta T \end{cases} \]

Condition for Static Force Balance:
\[ \frac{2\bar{P}}{2\varepsilon} = -p_0 g \left[ 1 - \alpha \left( T_0 - \frac{\Delta T}{d} \left( z + \frac{d}{2} \right) - T_0 \right) \right] \]
\[ = -p_0 g \left[ 1 + \alpha \frac{\Delta T}{d} \left( z + \frac{d}{2} \right) \right] \]

\[ z = \frac{d}{2} \quad \text{cooler} \]
\[ \bar{P}(z) \]

- Gravity only
- Gravity + thermal density change

z = 0

HOTTER
Nonlinear Equations for Perturbations

Subtract hydrostatic condition from equations for fluid dynamics to get nonlinear equations for perturbations.

\[
\begin{align*}
\frac{2U_x}{2t} + (\mathbf{U} \cdot \nabla) U_x &= -\frac{1}{\rho_0} \frac{2\rho}{2x} + g x \frac{\nabla^2}{\nabla^2 x} \\
\frac{2U_y}{2t} + (\mathbf{U} \cdot \nabla) U_y &= -\frac{1}{\rho_0} \frac{2\rho}{2y} + \nu \nabla^2 U_y \\
\frac{2U_z}{2t} + (\mathbf{U} \cdot \nabla) U_z &= -\frac{1}{\rho_0} \frac{2\rho}{2z} + \nu \nabla^2 U_z
\end{align*}
\]

\[
\frac{2u_z}{2t} + \frac{2u_y}{2y} + \frac{2u_x}{2x} = 0
\]

\[
\frac{2\tilde{T}}{2t} + (\mathbf{U} \cdot \nabla)(\tilde{T}(t) + \nabla^2) = \kappa \nabla^2 \tilde{T}
\]

Unknown functions: \(U_x, U_y, U_z, \rho, \tilde{T}\)

5 equations + 5 unknowns
Linear Equations

(A) \[ \frac{2u_z}{2t} = -\frac{1}{\rho_0} \frac{2\rho}{\delta z} + g \alpha \frac{\Gamma}{T} + \gamma \nabla^2 u_z \]

(B) \[ \frac{2u_x}{2t} = -\frac{1}{\rho_0} \frac{2\rho}{\delta x} + \gamma \nabla^2 u_x \]

(C) \[ \frac{2u_y}{2t} = -\frac{1}{\rho_0} \frac{2\rho}{\delta y} + \gamma \nabla^2 u_y \]

(D) \[ \frac{2u_x}{2x} + \frac{2u_y}{2y} + \frac{2u_z}{2z} = 0 \]

(E) \[ \frac{2T}{2t} + u_z \frac{2T}{2z} = \kappa \nabla^2 T \]

\[ \Gamma = -\frac{\Delta T}{d} \]

Plan for simplification:

- RELATE \( \hat{\rho} \) AND \( \hat{T} \) BY COMBINING Eqs A, B, C, D
- ELIMINATE \( \hat{\rho} \) FROM A BY USING RELATIONSHIP ABOVE
- SOLVE TWO SIMULTANEOUS PDES FOR \( \tilde{U}_2 \) AND \( \tilde{T} \)
Simplification

- Combine A, B, C, D ...

\[ \frac{2}{\rho_0} \frac{\partial^2 \tilde{P}}{\partial t^2} \left( \frac{2u_x}{2x} + \frac{2u_y}{2y} + \frac{2u_z}{2z} \right) \right) = -\frac{1}{\rho_0} \left( \frac{2^3 \tilde{P}}{2x^3} + \frac{2^3 \tilde{P}}{2y^3} + \frac{2^3 \tilde{P}}{2z^3} \right) + \rho \frac{\partial^2 \tilde{P}}{\partial t^2} + \gamma \nabla^2 (\nabla \cdot \tilde{U}) \]

\[ \nabla \cdot \tilde{U} = 0, \text{ so} \]

\[ \frac{1}{\rho_0} \frac{\partial^2 \tilde{P}}{\partial t^2} = \rho \frac{\partial^2 \tilde{P}}{\partial t^2} \]

- Take Laplacian of A ...

\[ \frac{2}{\rho_0} \frac{\partial^2 u_2}{\partial t^2} = -\frac{1}{\rho_0} \frac{2}{2z} \nabla^2 \tilde{P} + \rho \frac{\partial^2 \tilde{P}}{\partial t^2} + \gamma \nabla^2 \nabla^2 u_2 \]

- Eliminate \( \tilde{P} \) ...

\[ \frac{2}{\rho_0} \frac{\partial^2 u_2}{\partial t^2} = \rho \left( \nabla^2 \tilde{T} - \frac{2^2 \tilde{T}}{2z^2} \right) + \gamma \nabla^2 \nabla^2 u_2 \]
Two Coupled PDEs forms a Linear Eigensystem

\[ \frac{2}{\alpha} \nabla^2 u_+ - \gamma \nabla^2 \nabla^2 u_+ = \alpha \left( \nabla^2 - \frac{\alpha^2}{2z^2} \right) \tilde{T} \]

\[ \frac{2}{\alpha} \tilde{T} - \kappa \nabla^2 \tilde{T} = \pm u_+ \Gamma \]

Once \( u_+ (x, y, z, t) \) and \( \tilde{T} (x, y, z, t) \) are known, we can find the forms for \( (\tilde{\phi}, u_x, u_y) \).
Boundary Conditions

\[ t = \pm \frac{d}{2} \]

\[ \bar{u} = 0 \text{ at } t = \pm \frac{d}{2} \] (T set by plates, constant)

\[ (\bar{u}_x, \bar{u}_y, \bar{u}_z) = 0 \text{ at } t = \pm \frac{d}{2} \] (no slip boundaries)

Since \((\bar{u}_x, \bar{u}_y) = 0 \) everywhere at boundary...

Then \( \nabla \cdot \bar{u} = 0 \) implies \( \frac{\partial \bar{u}_z}{\partial t} = -\left( \frac{2\bar{u}_x}{2x} + \frac{2\bar{u}_y}{2y} \right) = 0 \)
Normal Modes: Rewrite PDE as ODE

\[
\text{LET } \quad U_\varphi(x, y, z, t) = \mathcal{F}_\varphi(t) e^{st} e^{i(b_x x + b_y y) + k_z z}
\]

\[
\hat{F}(x, y, z, t) = \hat{F}(t) e^{st} e^{i(b_x x + b_y y) + k_z z}
\]

\[
b^2 = b_x^2 + b_y^2
\]

\text{NOTE: } \quad \text{IF } R_0(s) > 0, \quad \underline{\text{UNSTABLE}} \text{ growing modes}

\text{IF } R_0(s) < 0, \quad \underline{\text{Damped, STABLE}} \text{ modes}

\text{IF } \Im(s) \neq 0, \text{ modes are oscillatory, "wave-like"}

\text{NOTE: } \quad \frac{1}{2t} \to s

\nabla^2 \to \frac{\partial^2}{\partial z^2} - b^2

\text{PDE}(x, y, z, t) \to \text{ODE}(z)
Mode Equations

Perturbed

Vertical

Velocity :

\[ \mathbf{u}_2 \left( \frac{\partial^2 \mathbf{u}_2}{\partial x^2} - \frac{h^2}{2} \frac{\partial^2 \mathbf{u}_2}{\partial z^2} \right) - \nabla \left( \frac{\partial^2 \mathbf{u}_2}{\partial z^2} - \frac{h^2}{2} \frac{\partial^2 \mathbf{u}_2}{\partial z^2} \right) \mathbf{u}_2 = -9 \alpha \frac{h^2}{2} \mathbf{T} \]

Perturbed

Thermal :

\[ \frac{\partial T}{\partial t} - \kappa \left( \frac{\partial^2 T}{\partial x^2} - \frac{h^2}{2} \frac{\partial^2 T}{\partial z^2} \right) = + \mathbf{u}_2 \cdot \mathbf{\nabla} T \]

A more convenient form:

\[ \left( \frac{\partial^2}{\partial x^2} - \frac{h^2}{2} \frac{\partial^2}{\partial z^2} \right) \mathbf{u}_2 - \frac{s}{\nu} \left( \frac{\partial^2}{\partial x^2} - \frac{h^2}{2} \frac{\partial^2}{\partial z^2} \right) \mathbf{u}_2 = \frac{9 \alpha}{\nu} \frac{h^2}{2} \frac{\partial T}{\partial z} \]

\[ \frac{\partial^2 T}{\partial z^2} - \left( \frac{\partial^2}{\partial x^2} + \frac{5}{\kappa} \right) T = -\frac{s}{\kappa} \mathbf{u}_2 \]

Need to:

- Find form of \( \mathbf{u}_2(t) \) that satisfies boundary (i.e. modes)
- Find 'Eigenvalues' or relationship between \( s \) and \( h^2 \) and mode structure
Dimensionless Form

length $\sim d$

time $\sim d^2/k$ (the thermal diffusion time scale)

temperature $\sim \Delta T$

velocity $\sim d/(d^2/k) \sim (K/d)$

Then, 

$$\frac{\theta^2}{2d^2} \rightarrow \frac{\Delta T}{d^2} \frac{\theta^2}{2d^2}$$

$\uparrow$ Dimensionless Form

$$(\frac{K}{d} + \frac{\theta}{K}) \rightarrow \frac{\Delta T}{d^2} \left( \frac{K}{d} + \frac{\theta}{K} \right)$$

$\uparrow$ Dimensionless Form

$$\frac{\theta d}{K} \rightarrow \frac{\Delta T}{d} \left( \frac{K}{d} \right) \uparrow \frac{\Delta T}{d^2} \uparrow$$

$\uparrow$ Dimensionless Form

ETC
Dimensionless Equations

\[
\left[ \left( \frac{2^3}{2^2} - \frac{h^2}{b^2} \right)^2 - \frac{5}{P_R} \left( \frac{2^3}{2^2} - \frac{h^2}{b^2} \right) \right] \hat{U}_2 = R_a \frac{h^2}{b^2} \hat{T}
\]

\[
\frac{2 \hat{T}}{2 \frac{T}{2}} - (h^2 + 5) \frac{\hat{T}}{T} = -\hat{U}_2
\]

Where \( P_R \equiv \frac{Y}{\lambda} = \text{Prandtl Number} = \frac{\text{Momentum Diffusion (Viscosity)}}{\text{Heat Diffusion}} \)

\[
R_a = \frac{g \alpha \Delta T b^3}{\lambda} = \text{Rayleigh Number} = \frac{\text{Boudary Force}}{\text{Diffusion Force}}
\]

**Only two numbers parameterize dynamical behavior!**

**Boundaries:**

\[
\hat{T} \left( z = \pm \frac{1}{2} \right) = 0
\]

\[
\hat{U}_2 \left( z = \pm \frac{1}{2} \right) = 0
\]

\[
2 \frac{\hat{U}_2}{\hat{T}} \left( z = \pm \frac{1}{2} \right) = 0
\]
Marginal Instability/Stability

Let $S = 0$, find relationship between modes and parameters ($Re$, $Pr$) for threshold for instability.

If $S = 0$, then

\[
\left( \frac{2^3}{2^2} - \frac{1}{2} \right)^2 \hat{u}_z = Re \, \frac{1}{2} \hat{u}_z
\]

\[
\left( \frac{2}{2^2} - \frac{1}{2} \right) \frac{1}{1} = -\hat{u}_z
\]

On

\[
\left( \frac{2^2}{2^3} - \frac{1}{2} \right)^3 \hat{u}_z = -Re \, \frac{1}{2} \hat{u}_z
\]

\[
\left( \frac{2^2}{2^3} - \frac{1}{2} \right) \frac{1}{1} = -\hat{u}_z
\]

**NOTE:**
- $q$ is complex
- 6 roots for $q$
- Because Eq for $U_2$ is 6th-order ODE!
Solution for Marginal Instability/Stability

**Even mode**

\[
\hat{U}_x = \cos(qz)
\]

\[
\frac{2^2U_x}{2^2} = -q^2 \cos(qz)
\]

**Odd mode**

\[
\hat{U}_x = \sin(qz)
\]

\[
\frac{2^2U_x}{2^2} = -q^2 \sin(qz)
\]

**Yields great simplification...**

\[
(q^2 + l^2)^2 = \frac{Re \cdot l^2}{R_e} = \frac{9\alpha^4 \delta d^3}{\gamma K}
\]

**Six roots for \( q ...**

\[
q^2 = -l^2 + (1)^{1/3} \sqrt[3]{Re \cdot l^2}
\]

**Where**

\[
(1)^{1/3} = \left\{ \begin{array}{l}
1^{1/3} = 1 \\
i \pi/3 = -\frac{1}{2} (1 + \sqrt{3}i) \\
i \pi/3 = -\frac{1}{2} (1 - \sqrt{3}i)
\end{array} \right.
\]
Bénard Photos
Fun 2D Simulation Shows Linear to Nonlinear Evolution
Bénard Photos

Benard-Maragoni
Solar Granulation

Photospheric granulation, G. Scharmer
Swedish Vacuum Solar Telescope
10 July 1997
Mathematica Notebook

Bernard-Marginal-Mode.nb
Summary

- Instabilities are (initially) small perturbations that grow in time.
- The method used to find the mode structure and growth time for a linear instability is similar to the method used to find wave structure and dispersion.
- Bénard-Rayleigh thermal instability occurs when the temperature gradient exceeds a threshold.
- As the $\Delta T$ increases, a single mode becomes unstable first.