APPH 4200
Physics of Fluids

More Fluid Waves (Ch. 7)

1. Phasors
2. Surface Tension
3. Capillary Waves
4. Rayleigh-Taylor Instability
5. Sound
6. Organ Pipes
7. HW 4

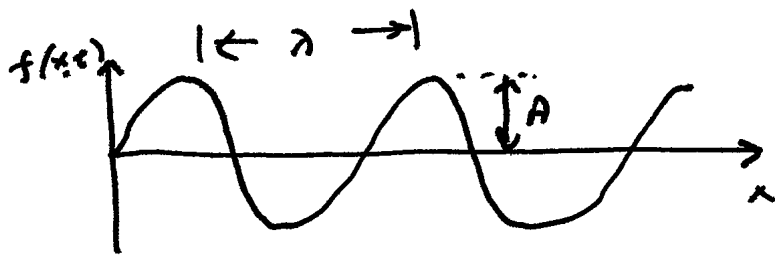
Plane Waves

$$f(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x \pm c t)\right)$$

AMPLITUDE

WAVELENGTH

SPEED



FREQUENCY $= \omega = 2\pi \frac{c}{\lambda}$ RAD/SEC

$\left(\frac{c}{\lambda}\right)$ = FREQUENCY IN HERTZ

$\varphi(x,t) \equiv \frac{2\pi}{\lambda}(x \pm c t)$ = PHASE OF WAVE

"PHASE FRONTS" ARE SURFACES OF CONSTANT PHASE

"PHASE FRONTS" MOVE AT SPEED c

c = PHASE VELOCITY

Linearized Gravity Waves

$$\nabla \cdot \bar{u} = 0 \implies \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0$$

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} - \nabla p \implies \begin{cases} \rho_0 \dfrac{\partial u_z}{\partial t} = -\rho_1 g - \dfrac{\partial p_1}{\partial z} \\ \rho_0 \dfrac{\partial u_x}{\partial t} = -\dfrac{\partial p_1}{\partial x} \end{cases}$$

$$\frac{\partial \rho}{\partial t} + \bar{u} \cdot \nabla \rho = 0 \implies \frac{\partial \rho_1}{\partial t} + u_z \frac{\partial \rho_0}{\partial z} = 0$$

EACH EQUATION CONTAINS TERMS THAT ARE <u>LINEAR</u> IN THE WAVE AMPLITUDE

<u>EASIEST</u> SOLUTION METHOD:

USE <u>PLANE-WAVE</u> SOLUTIONS THAT VARY AS

$$e^{-j(\omega t - k_x x - k_z z)}$$

THEN

FOR EXAMPLE

$$u_x(x,z,t) = \mathrm{Re}\left\{ u_x e^{-j(\omega t - k_x x - k_z z)} \right\}$$

COMPLEX AMPLITUDE INCLUDES PHASE

Plane-Wave Equations for Complex Wave Amplitude

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \implies k_x u_x + k_z u_z = 0$$

$$\rho_0 \frac{\partial u_z}{\partial t} = -\rho_1 g - \frac{\partial p_1}{\partial z} \implies -j\omega\rho_0 u_z = -\rho_1 g - j k_z p_1$$

$$\rho_0 \frac{\partial u_x}{\partial t} = -\frac{\partial p_1}{\partial x} \implies -j\omega\rho_0 u_x = -j k_x p_1$$

$$\frac{\partial \rho_1}{\partial t} + u_z \frac{\partial \rho_0}{\partial z} = 0 \implies -j\omega\rho_1 + u_z \frac{\partial \rho_0}{\partial z} = 0$$

$$\left.\begin{aligned} \rho_1 &= -j\frac{u_z}{\omega}\frac{\partial \rho_0}{\partial z} \\ p_1 &= \frac{\omega}{k_x}\rho_0 u_x \\ u_x &= -\frac{k_z}{k_x} u_z \end{aligned}\right\}$$

AN EQUATION FOR u_z ...

$$-j\omega\rho_0 u_z = j u_z \frac{g}{\omega}\frac{\partial \rho_0}{\partial z} - j\omega\frac{k_z}{k_x}\rho_0\left(-\frac{k_z}{k_x}u_z\right)$$

PRODUCES A DISPERSION RELATION

$$-\omega = \frac{g}{\omega}\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} + \omega\left(\frac{k_z}{k_x}\right)^2$$

OR

$$\boxed{\omega^2\left(1 + \left(\frac{k_z}{k_x}\right)^2\right) = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}}$$

$$\Omega_B^2 \equiv -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$$

"BUOYANCY FREQUENCY"

Phasors, Sine, Cosine

$$\sin(kx-\omega t) = \mathrm{Re}\left\{ e^{i\frac{\pi}{2}}\, e^{-j(\omega t - kx)} \right\} \qquad \text{PHASOR} = e^{i\pi/2}$$

$$\cos(kx-\omega t) = \mathrm{Re}\left\{ e^{-j(\omega t - kx)} \right\} \qquad \text{PHASOR} = 1$$

$$\cos(kx-\omega t+\theta) = \mathrm{Re}\left\{ e^{i\theta}\, e^{-j(\omega t - kx)} \right\} \qquad \text{PHASOR} = e^{i\theta}$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos(kx-\omega t) = \int_0^\lambda \frac{dx}{\lambda} \sin(kx-\omega t) = 0$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos^2(kx-\omega t) = \int_0^{2\pi} \frac{d\xi}{2\pi} \cos^2(\xi-\omega t) = \frac{1}{2}$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos(kx-\omega t)\cos(kx-\omega t+\theta) = \frac{1}{2}\cos(\theta)$$

$$\int_0^\lambda \frac{dx}{\lambda} \cos(kx-\omega t)\sin(kx-\omega t) = 0$$

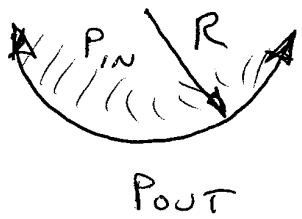
IF A = PHASOR, THEN

$\frac{1}{2}|A|^2 = \frac{1}{2}A^*A$ = SQUARE AVERAGE OF $\mathrm{Re}\left\{A e^{-j(\omega t - kx)}\right\}$

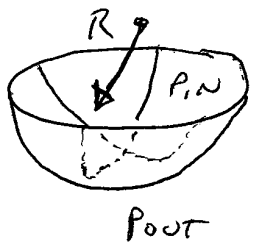
$|A|/\sqrt{2} \equiv$ RMS OF A

Surface Tension

LAPLACE'S FORMULA FOR SURFACE TENSION



$$P_{IN} - P_{OUT} = \frac{\sigma}{R} \quad \text{(2D "TROUGH")}$$



$$P_{IN} - P_{OUT} = \frac{2\sigma}{R} \quad \text{(SPHERICAL BUCKET)}$$

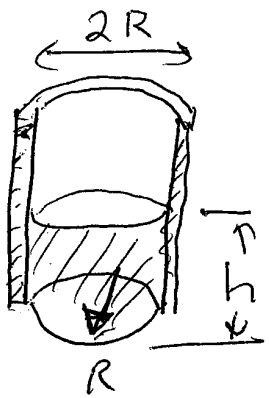
$\sigma \equiv$ SURFACE TENSION

$= 0.074\ N/m$ FOR AIR-WATER

SURFACE TENSION "PULLS" ON SURFACE IN ORDER TO MAKE IT MINIMIZE FREE ENERGY OF INTERFACE.

SURFACE TENSION $\begin{cases} \text{MAKES DROPLETS ROUND} \\ \text{MAKES RIPPLES FLAT} \end{cases}$

Example: Water Suspended in a Capillary Tube



$$P_{IN} - P_{OUT} = \frac{2\sigma}{R}$$

BUT $P_{IN} = \rho g h \gg P_{OUT}$

SO $h = \frac{2\sigma}{\rho g}\frac{1}{R} = \frac{a^2}{R}$ = MAXIMUM HEIGHT OF SUSPENDED WATER IN CAPILLARY

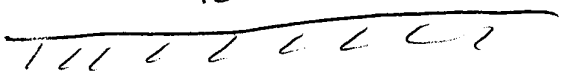
$$\sqrt{\frac{2\sigma}{\rho g}} \equiv a \equiv \text{CAPILLARY CONSTANT}$$

$$\left.\begin{aligned} \sigma &= 0.074\ N/m \\ \rho &= 1000\ kg/m^3 \text{ WATER} \\ g &= 9.8\ m/s^2 \end{aligned}\right\} \quad a = 3.9\ \text{mm}$$

Gravity Waves with Surface Tension

$$\eta(x,t) = \eta_0 \cos(kx - \omega t)$$

$$\frac{1}{R} = \frac{\partial^2 \eta}{\partial x^2}$$



SURFACE TENSION ACTS LIKE A RESTORING FORCE...

PULLS ON SURFACE TO MAKE $\frac{\partial \eta}{\partial x} \to 0$

SURFACE WAVE:

$$\bar{u} = \nabla \varphi \qquad \nabla^2 \varphi = 0 \qquad \varphi(x,z,t) = f(z)\sin(kx - \omega t)$$

BERNOULLI:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}u^2 + \frac{(P_{IN} - P_{OUT})}{\rho} + g\eta = 0 \quad \text{AT SURFACE}$$

LINEAR PART: $\boxed{\frac{\partial \varphi}{\partial t} = \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} - g\eta}$ AT $z=0$

SURFACE CONSTRAINT $\boxed{\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t}}$ AT $z=0$

Gravity/Capillary Wave

$$\nabla^2\varphi = 0 \Rightarrow \quad f'' - k^2 f = 0 \quad \text{or} \quad f(z) \propto e^{kz} \quad (\text{FOR } z<0)$$

SURFACE BOUNDARY CONDITIONS:

$$\frac{\partial\varphi}{\partial t} = \frac{\sigma}{\rho}\frac{\partial^2\eta}{\partial x^2} - g\eta \Rightarrow -\omega f = -\left(g + \frac{\sigma}{\rho}k^2\right)\eta_0$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t} \Rightarrow \underbrace{kf = \omega\eta_0}$$

$$\omega^2 = k\left(g + \frac{\sigma}{\rho}k^2\right)$$ DISPERSION RELATION

WHEN $g \gg \frac{\sigma}{\rho}k^2$ (OR $\lambda^2 \gg 4\pi^2\frac{\sigma}{\rho g}$) $\sim \lambda \gg 2\pi a$

a: CAPILLARY CONSTANT

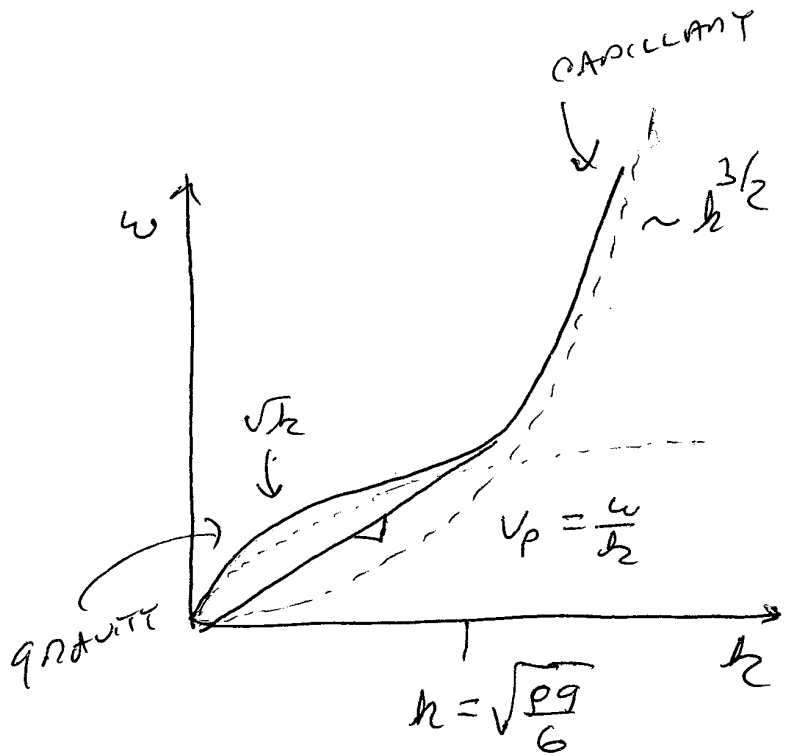
THEN <u>SURFACE TENSION</u> HAS NO EFFECT

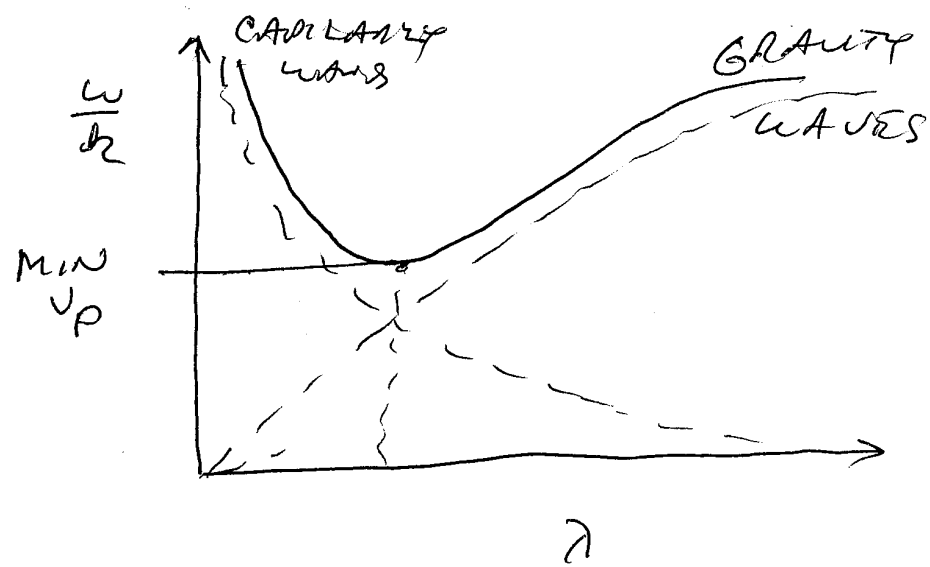
WHEN $g \ll \frac{\sigma}{\rho}k^2$ THEN WE HAVE "CAPILLARY WAVES"

$$\omega^2 = \frac{\sigma}{\rho}k^3$$

AND <u>GRAVITY</u> HAS NO EFFECT

Gravity/Capillary Dispersion





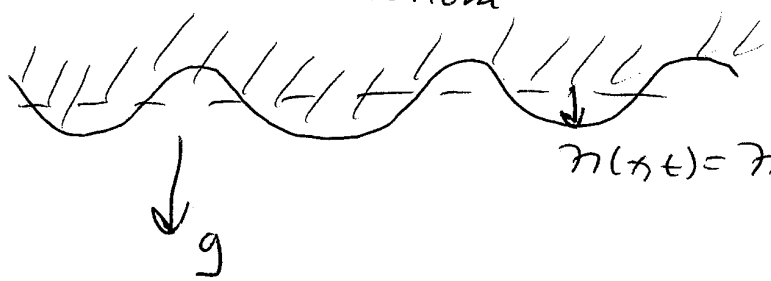
$$\omega = \sqrt{kg + \frac{\sigma}{\rho}k^3}$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{2\pi}\lambda + \frac{\sigma}{\rho}\frac{2\pi}{\lambda}}$$

Heavy Fluid on Top of Light Fluid

(RAYLEIGH-TAYLOR INSTABILITY)

$\rho_{TOP} \gg \rho_{BOTTOM}$



$\eta(x,t) = \eta_0 \cos(kx - \omega t)$

GRAVITY WAVE WITH REVERSED DENSITY GRADIENT OR
REVERSED DIRECTION OF GRAVITY

$\bar{U} = \nabla \varphi \qquad \nabla^2 \varphi = 0 \qquad \varphi(x,z,t) = f(z)\sin(kx - \omega t)$

WITH $f(z) \sim e^{-kz} \quad (z > 0)$

SURFACE CONDITIONS:

$$\left.\begin{aligned} -\frac{\partial \varphi}{\partial z} &= \frac{\partial \eta}{\partial t} \\ \frac{\partial \varphi}{\partial t} &= +g\eta \end{aligned}\right\} \qquad \begin{aligned} kf &= \omega \eta_0 \\ -\omega f &= g\eta_0 \end{aligned}$$

$\uparrow$ ~~XXX~~

$$e^{-j\omega t} \Rightarrow e^{\sqrt{gk}\,t}$$

$$\omega^2 = -gk \Rightarrow \underline{\underline{\omega = \pm i\sqrt{gk}}}$$

INSTABILITY

Rayleigh-Taylor with Surface Tension

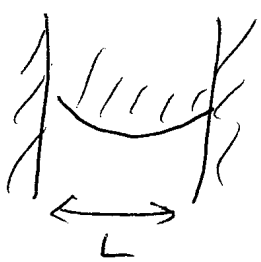
WHAT IS UNSTABLE GROWTH RATE WITH SURFACE TENSION?

$$\frac{\partial \varphi}{\partial t} = \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} + g\eta$$

$$-\frac{\partial \varphi}{\partial t} = \frac{\partial \eta}{\partial t}$$

$$\omega^2 = -k\left(g - \frac{\sigma}{\rho}k^2\right)$$

IF $\frac{\sigma}{\rho}k^2 > g$, THEN RAYLEIGH-TAYLOR INSTABILITY IS STABILIZED!



$$\frac{\sigma}{\rho}\left(\frac{\pi}{L}\right)^2 > g \quad \text{OR} \quad L < \pi\sqrt{\frac{\sigma}{\rho g}}$$

Sound

SOUND WAVES INVOLVE COMPRESSION OF FLUID.

EQUATIONS REQUIRED:

- CONSERVATION OF MASS

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{u} = 0$$

- EULER'S EQUATION (MOMENTUM / FORCE)

$$\rho \frac{D\bar{u}}{Dt} = -\nabla p$$

- EQUATION OF STATE

SOUND WAVES ARE (USUALLY) ADIABATIC.

$p(\rho)$ FOR EXAMPLE: $p/\rho^{\gamma} = \text{CONSTANT}$

WITH γ = RATIO OF SPECIFIC HEATS

LINEARIZED FIELD VARIABLES:

$$\rho = \rho_0 + \rho_1$$
$$p = p_0 + p_1$$

$\bar{u}$ ~ ACCOUSTIC FLOW OSCILLATIONS

Linearized Sound Waves

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \bar{u} = 0$$

$$\rho_0 \frac{\partial \bar{u}}{\partial t} = -\nabla P_1$$

WITH $P(\rho)$ ADIABATIC

$$P_1 = \left.\frac{\partial P}{\partial \rho}\right|_0 \rho_1$$

$$\left.\frac{\partial P}{\partial \rho}\right|_0 \equiv c_s^2 = \text{SQUARE OF SPEED OF SOUND}$$

WITH $P/\rho^\gamma = \text{CONSTANT}$ $\quad c_s^2 \equiv \gamma \left(\frac{P}{\rho}\right)_0$

$$\rho_0 \frac{\partial^2 \bar{u}}{\partial t^2} = -\nabla \frac{\partial P_1}{\partial t} = -\nabla\left(c_s^2 \frac{\partial \rho_1}{\partial t}\right)$$

$$= \nabla\left(c_s^2 \rho_0 \nabla \cdot \bar{u}\right)$$

IF $\nabla \times \bar{u} = 0$ (AS IT IS FOR USUAL SOUND WAVES), THEN

$$\bar{u} = \nabla \varphi$$

AND $\boxed{\rho_0 \frac{\partial^2 \varphi}{\partial t^2} = \rho_0 c_s^2 \nabla^2 \varphi}$ ← HELMHOLTZ EQUATION!

[NOTE: WITH (P_0, ρ_0) UNIFORM, THEN c_s = CONSTANT. LINEAR SOUND IS NON-DISPERSIVE!]

What is the energy density of a sound wAve?

ENERGY OF FLUID $= \rho E + \frac{1}{2}\rho u^2$

$\uparrow$ THERMAL INTERNAL ENERGY

$\uparrow$ MECHANICAL ENERGY

WAVE ENERGY DENSITY = AVG ENERGY ASSOCIATED WITH CHANGES TO INTERNAL ENERGY + AVG MECHANICAL ENERGY

WAVE MECHANICAL ENERGY $= \frac{1}{2}\rho_0 \overline{u^2}$

WAVE INTERNAL ENERGY IS ASSOCIATED WITH OSCILLATING "COMPRESSION"

$$(\rho E) \approx \rho_0 E_0 + \rho' \left.\frac{\partial(\rho E)}{\partial \rho}\right|_0 + \frac{1}{2}\rho'^2 \left.\frac{\partial^2(\rho E)}{\partial \rho}\right|_0 + \cdots$$

$\uparrow$ THIS IS UNPERTURBED ENERGY DENSITY

$\uparrow$ THIS WILL AVG TO ZERO

$\uparrow$ THIS IS WAVE INTERNAL ENERGY

FOR AN ADIABATIC FLUID, ENTROPY IS CONSTANT DURING SOUND OSCILLATIONS

AND $\delta E = -P\delta V = \frac{P}{\rho^2}\delta\rho$, SO $\frac{\partial(\rho E)}{\partial \rho} = E + \frac{P}{\rho}$, $\frac{\partial^2(\rho E)}{\partial \rho} = \frac{\partial P}{\partial \rho}\frac{\partial^2(\rho E)}{\partial P \partial \rho}$

$= c_s^2/\rho$

Sound Wave Energy Density

WAVE ENERGY DENSITY $= \frac{1}{2}\rho'^2 c_s^2/\rho_0 + \frac{1}{2}\rho_0 u^2$

BUT $\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla\cdot\bar{u}$, SO $\rho' = \frac{k}{\omega} u \rho_0$ SO $(\rho')^2 = \rho_0^2 c_s^{-2} u^2$

THEREFORE, INTERNAL (COMPRESSIVE) WAVE ENERGY DENSITY
<u>EQUALS</u> MECHANICAL ENERGY DENSITY

CONSERVATION OF WAVE ENERGY DENSITY

$$\rho_0 \frac{\partial \bar{u}}{\partial t} = -\nabla p' \Rightarrow \rho_0 \frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) = -\bar{u}\cdot\nabla p'$$
$$= -\nabla\cdot(p'\bar{u}) + p'\nabla\cdot\bar{u}$$

BUT $\nabla\cdot\bar{u} = -\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t}$ SO

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u^2\right) + \frac{p'}{\rho_0}\frac{\partial \rho'}{\partial t} = -\nabla\cdot(p'\bar{u})$$

USING $p' = c_s^2 \rho'$...

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u^2 + \frac{1}{2}\rho_1^2 c_s^2/\rho_0\right) = -\nabla\cdot(p'\bar{u})$$

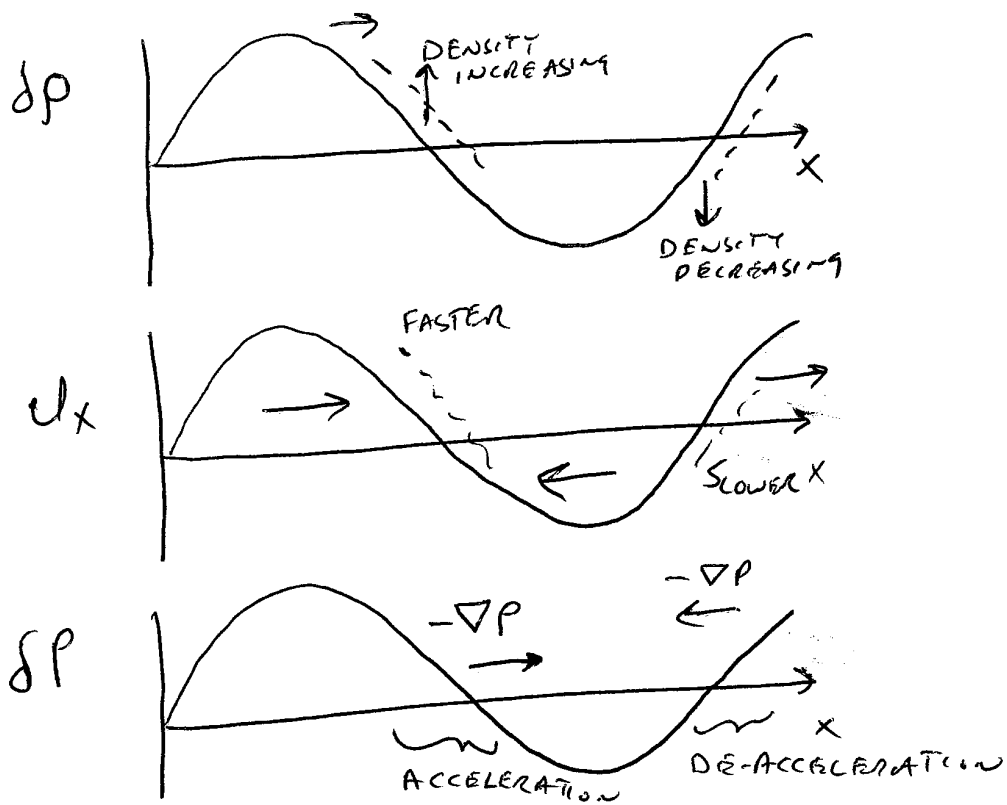
WHERE $p'\bar{u} = c_s^2 \frac{\rho'^2}{\rho_0}\left(\frac{\omega}{k}\right) = V_g$ (WAVE ENERGY DENSITY) !!

Close-up of Sound

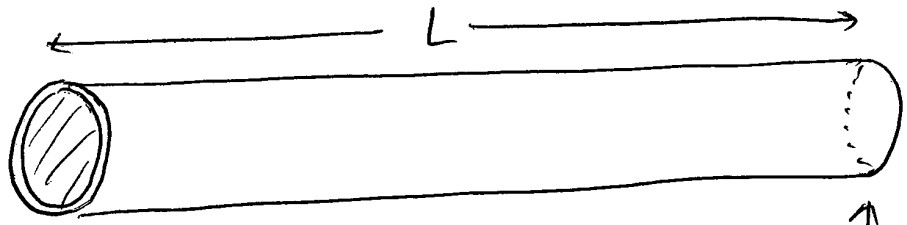
LET $\rho'(x,t) = \delta\rho \sin(kx - \omega t)$, THEN

$$u_x(x,t) = \left(\frac{\delta\rho}{\rho_0}\right)\left(\frac{\omega}{k}\right)\sin(kx - \omega t) \qquad \text{SINCE } \frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \vec{u}$$

$$p'(x,t) = c_s^2\, \delta\rho \sin(kx - \omega t)$$



Sound in a Pipe



MEMBRANE (SPEAKER)

$$u(t) = \mathrm{Re}\left\{ u_0 e^{-j\omega t} \right\}$$

OPEN END:

PRESSURE VARIATIONS MUST VANISH, BUT

SOUND WAVES CAUSE MODULATING FLOWS $u \neq 0$ AT $z = L$

SOLUTION:

$$u(x,t) = \left(\underbrace{C_1 e^{+jkx}}_{\text{FORWARD WAVE}} + \underbrace{C_2 e^{-jkx}}_{\text{BACKWARD WAVE}} \right) e^{-j\omega t}$$

EULER:

$$\frac{\partial p'}{\partial x} = -\rho_0 \frac{\partial u}{\partial t} = j\omega\rho_0 \left(C_1 e^{+jkx} + C_2 e^{-jkx} \right) e^{-j\omega t}$$

THUS

$$p'(x,t) = \frac{\omega}{k} \rho_0 \left(\underbrace{C_1 e^{+jkx}}_{\text{FORWARD}} - \underbrace{C_2 e^{-jkx}}_{\text{BACKWARD}} \right) e^{-j\omega t}$$

Matching Tube Boundaries

AT MEMBRANE: $u(x=0,t) = u_0(t)$

$$\boxed{C_1 + C_2 = u_0}$$

AT OPEN END: $p(x=L,t) = 0$

$$\boxed{C_1 e^{jkL} - C_2 e^{-jkL} = 0}$$

THIS MEANS

$$C_1 = \frac{u_0}{2\cos kL} e^{-jkL} \qquad C_2 = C_1 e^{j2kL} = \frac{u_0}{2\cos kL} e^{+jkL}$$

THEREFORE:

$$u(x,t) = \frac{u_0}{2\cos kL}\left(e^{jk(x-L)} + e^{-jk(x-L)}\right)$$

$$= u_0 \frac{\cos(k(x-L))}{\cos(kL)}$$

AT $x=L$, FLUID VELOCITY IS <u>MUCH, MUCH</u> LARGER AT RESONANCE!

A TUBE IS A <u>SOUND AMPLIFIER</u>!

HW 1: Ch 7, Prob 1

1. Consider stationary surface gravity waves in a rectangular container of length L and breadth b, containing water of undisturbed depth H. Show that the velocity potential

$$\phi = A\cos(m\pi x/L)\cos(n\pi y/b)\cosh k(z+H)\,e^{-i\omega t},$$

satisfies $\nabla^2\phi = 0$ and the wall boundary conditions, if

$$(m\pi/L)^2 + (n\pi/b)^2 = k^2.$$

Here m and n are integers. To satisfy the free surface boundary condition, show that the allowable frequencies must be

$$\omega^2 = gk\tanh kH.$$

[*Hint*: combine the two boundary conditions (7.27) and (7.32) into a single equation $\partial^2\phi/\partial t^2 = -g\,\partial\phi/\partial z$ at $z = 0$.]

HW 2: Ch 7, Prob 2

2. This is a continuation of Exercise 1. A lake has the following dimensions

$$L = 30\,\text{km} \quad b = 2\,\text{km} \quad H = 100\,\text{m}.$$

Suppose the relaxation of wind sets up the mode $m = 1$ and $n = 0$. Show that the period of the oscillation is 31.7 min.

HW 3: Ch 7, Prob 4

4. Plot the group velocity of surface gravity waves, including surface tension σ, as a function of λ. Assuming deep water, show that the group velocity is

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k}}\frac{1+3\sigma k^2/\rho g}{\sqrt{1+\sigma k^2/\rho g}}.$$

Show that this becomes minimum at a wavenumber given by

$$\frac{\sigma k^2}{\rho g} = \frac{2}{\sqrt{3}} - 1.$$

For water at 20 °C ($\rho = 1000\,\mathrm{kg/m^3}$ and $\sigma = 0.074\,\mathrm{N/m}$), verify that $c_{g\,\mathrm{min}} = 17.8\,\mathrm{cm/s}$.

Summary

- Surface tension acts to flatten short ripples on a fluid.
- Surface tension is a restoring force that can stabilize instability.
- Sound obeys Helmholtz Equation, with the speed of sound related to the $\sqrt{(\partial P/\partial \rho)}$.
- Organ pipes amplify resonant sound waves!