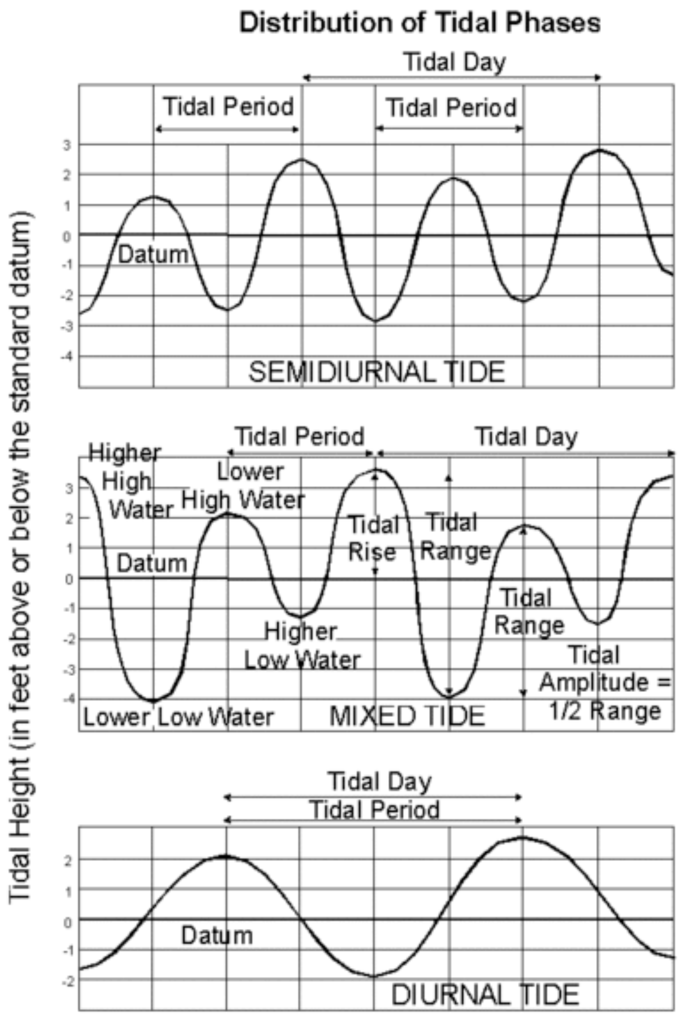
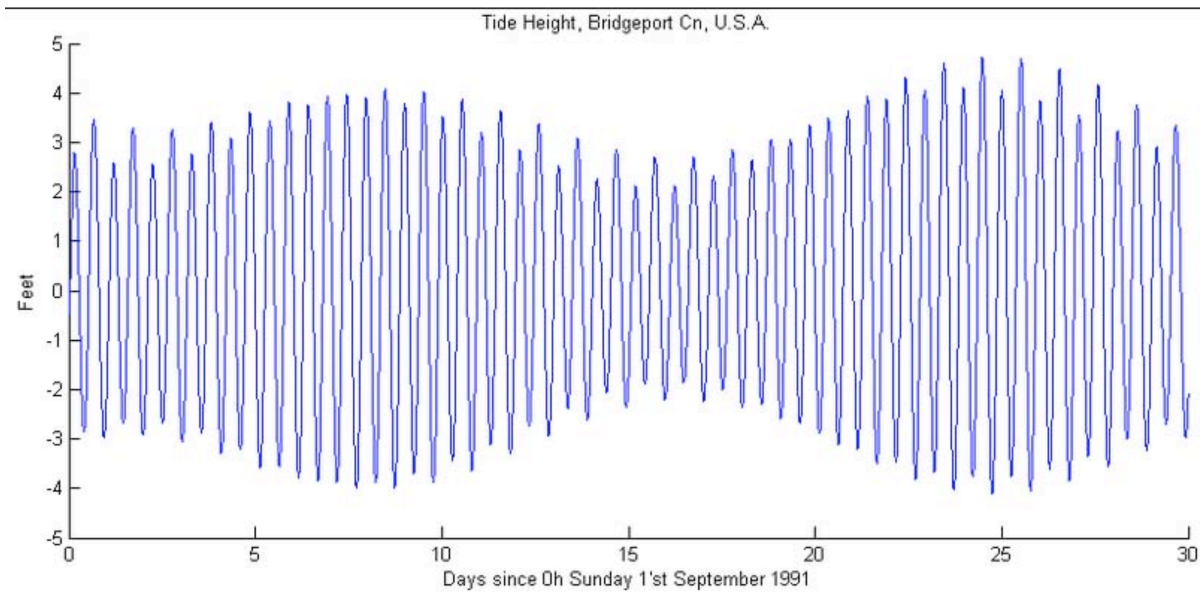
APPH 4200
Physics of Fluids

More Fluid Waves/Instabilities (Ch. 7)

1. Tides
2. Jeans instability

Tides





Thirty days of tide heights at Bridgeport Connecticut U.S.A. as calculated from the Harmonic Constituent data aligned with 0h Sunday 1'st September 1991.

10-component tide-predicting machine, conceived by Lord Kelvin

Tide predicting machine, by Sir William Thomson in 1876.

The first full sized machine for predicting tides, combined ten tidal components (one pulley for each component). It could trace the heights of the tides for one year in about four hours.

The M2 tidal constituent. Amplitude is indicated by color, and the white lines are cotidal differing by 1 hr. The curved arcs around the amphidromic points show the direction of the tides, each indicating a synchronized 6 hour period.

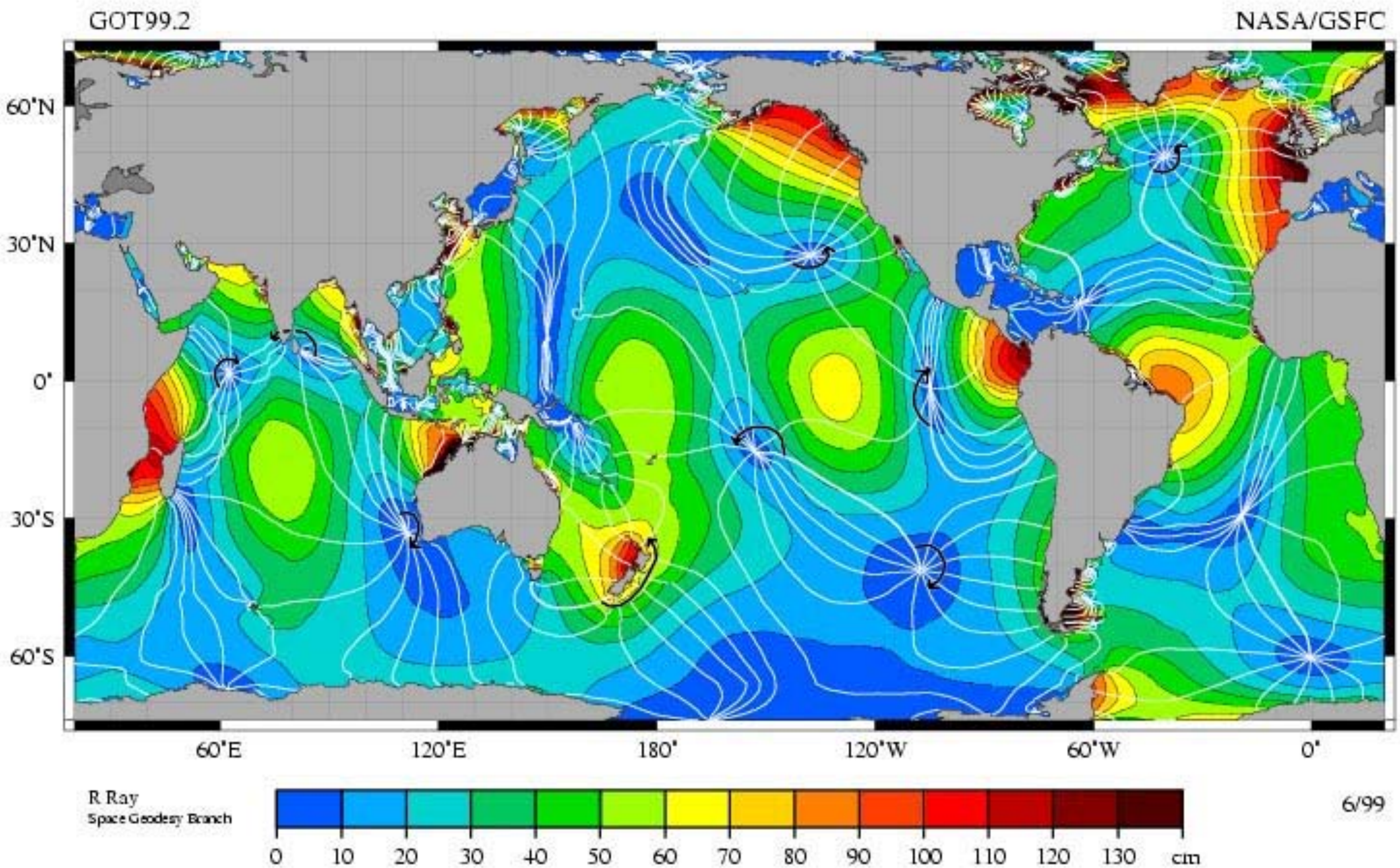
R. Ray, [TOPEX/Poseidon: Revealing Hidden Tidal Energy](#) [en:GSFC](#),[en:NASA](#). Redistribute with credit to R. Ray, and

NASA - Goddard Space Flight Center

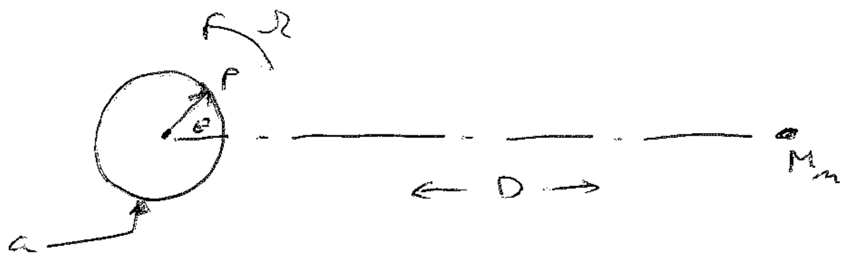
NASA - Jet Propulsion Laboratory

Scientific Visualization Studio

Semi-Diurnal Tide Map



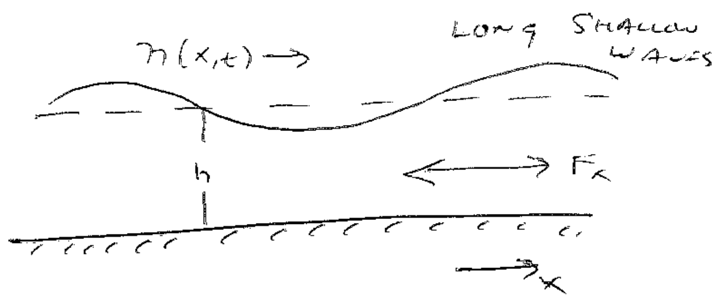
Tides: Driven Shallow Water Waves



EQUATIONS

$$\rho \frac{d\vec{v}}{dt} = -\nabla p - g\rho\hat{z} + \vec{F}$$

LUNAR DRIVE FORCE



LINEARIZE

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} - g\rho + F_z$$

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} + F_x$$

FOR SHALLOW WATER WAVES

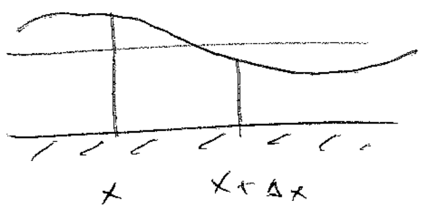
SO, LATERAL FORCING IS MOST IMPORTANT RELEVANT

$$\frac{|v_z|}{|v_x|} \sim kh \ll 1$$

AND $g\rho \gg |F_z|$

Surface Waves with a Lateral Force

Continuity



$$\rho h V_x(x) - \rho h V_x(x+\Delta x) = \rho \Delta x \frac{\partial \eta}{\partial t}$$

or

$$h \frac{\partial V_x}{\partial x} = -\frac{\partial \eta}{\partial t}$$

Euler (Horizontal)

$$\rho \frac{\partial V_x}{\partial t} = -\frac{\partial P}{\partial x} + F_x$$

Bernoulli (Linearized)

$$P(x) - P_0 = g \rho \eta \Rightarrow \frac{\partial P}{\partial x} = g \rho \frac{\partial \eta}{\partial x}$$

Collecting terms ...

$$\rho h \frac{\partial^2 V_x}{\partial x \partial t} = -\rho \frac{\partial^2 \eta}{\partial t^2}$$

$$\rho h \frac{\partial^2 V_x}{\partial x \partial t} = -\rho g h \frac{\partial^2 \eta}{\partial x^2} + h \frac{\partial F_x}{\partial x}$$

$$\frac{\partial^2 \eta}{\partial t^2} = g h \frac{\partial^2 \eta}{\partial x^2} - \frac{h}{\rho} \frac{\partial F_x}{\partial x} \quad \leftarrow **$$

(Same Eq using "integrated" velocity potential)

LET $\vec{u} = \nabla \varphi$ WITH $\nabla \cdot \vec{u} = \nabla^2 \varphi = 0$

BUT SHALLOW WATER ASSUMPTION IMPLIES $\left|\frac{\partial \varphi}{\partial z}\right| \ll \left|\frac{\partial \varphi}{\partial x}\right|$

LINEARIZED BERNOULLI

$$\frac{\partial \varphi}{\partial t} = -\frac{P}{\rho} - \frac{\Phi}{\rho} \qquad (\text{WITH } \vec{F} = -\nabla \Phi)$$

AND $\delta\left(P(x) - P_0 = g\rho\eta\right) = 0$ SO $\frac{\partial P}{\partial x} = g\rho\frac{\partial \eta}{\partial x}$

SURFACE CONSTRAINT

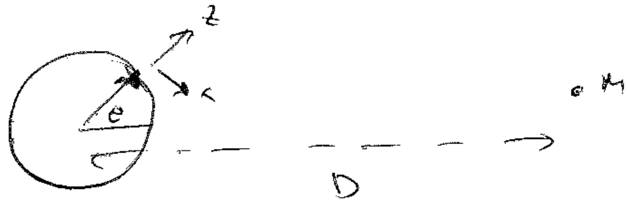
$$\frac{\partial \varphi}{\partial z} = \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial^2 \varphi}{\partial z^2} = -\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial \eta}{\partial t \partial z}$$

INTEGRATING VERTICALLY ACROSS THIN/SHALLOW FLUID $\Rightarrow$ $\frac{\partial \eta}{\partial t} = -h\frac{\partial^2 \varphi}{\partial x^2}$

SO

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \varphi}{\partial x^2}\right) = -g\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{\rho}\frac{\partial^2 \Phi}{\partial x^2} \Rightarrow \boxed{\frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial^2 \eta}{\partial x^2} - \frac{h}{\rho}\frac{\partial F_x}{\partial x}} \;\; **$$

What is the lateral force?



$$\frac{\partial F_x}{\partial x} = \frac{1}{a}\frac{\partial F_\theta}{\partial \theta}$$

$$\frac{F}{\rho} = -\nabla \Psi \qquad \Psi = \frac{GM_m}{\sqrt{D^2+a^2-2aD\cos\theta}} = \frac{GM_m}{D}\left(1+\left(\frac{a}{D}\right)^2-2\frac{a}{D}\cos\theta\right)^{1/2}$$

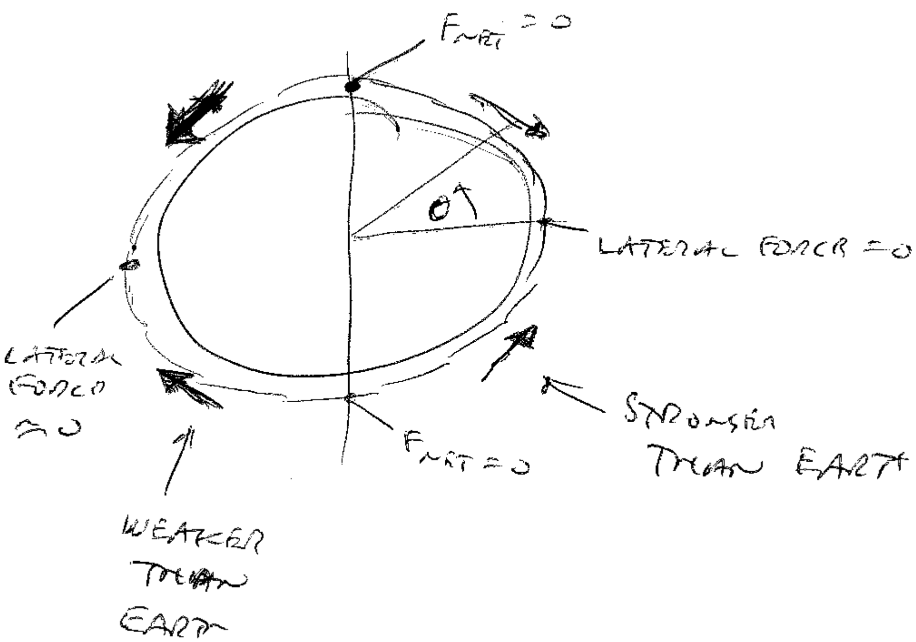
BUT LATERAL FORCE MUST BE <u>THE</u> <u>DIFFERENCE</u> BETWEEN THE FORCE ON THE FLUID AND THE FORCE ON THE WHOLE EARTH. THIS IS THE <u>TIDAL</u> <u>FORCE</u>.

$$\frac{F_{NET}}{\rho} = -\nabla\left[\frac{GM_m}{D}\left(1+\left(\frac{a}{D}\right)^2-2\frac{a}{D}\cos\theta\right)^{-1/2} - \frac{GM_m}{D}\left(1+\frac{a}{D}\cos\theta\right)\right]$$

$$(1+x)^{-1/2} \simeq 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

$$\frac{F_{NET}}{\rho} \simeq -\nabla\left[\frac{1}{2}\frac{GM_m}{D^3}a^2\left(1-3\cos^2\theta\right)\right] = \frac{3}{2}\frac{GM_m\, a}{D^3}\underbrace{2\sin\theta\cos\theta}_{\sin(2\theta)}$$

Net Force with Period 2Ω



M_m

$$\frac{F_{NET}}{\rho} = \frac{3}{2}\,\frac{GM_m}{D^3}\,a \sin 2\theta$$

Driven Surface Waves

$$\frac{\partial^2 \eta}{\partial t^2} = gh \frac{\partial^2 \eta}{\partial x^2} - h \frac{\partial}{\partial x}\left(\frac{F_{NET}}{\rho}\right)$$

$$\frac{\partial}{\partial x} = -\frac{1}{a}\frac{\partial}{\partial \theta} \qquad \text{AND} \qquad \theta \approx \Omega t \qquad \Omega = \frac{2\pi}{24\text{ hours}}$$

SO

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{gh}{a^2}\underbrace{\frac{\partial^2 \eta}{\partial \theta^2}} + 3\,\frac{G M_m}{D^3}\,h \cos 2\theta$$

$$\frac{\partial^2 \eta}{\partial \theta^2} = \frac{1}{\Omega^2}\frac{\partial^2 \eta}{\partial t^2} \qquad\qquad \frac{G M_E}{a^2} \sim g$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\frac{3\,G M_m h}{D^3\left(1 - \frac{gh}{a^2 \Omega^2}\right)} \cos 2\Omega t$$

$gh = 9.8 \times 4000 \sim 4 \times 10^4\ m^2/s^2$

$(a^2 \Omega^2) = \left(6 \times 10^6 \frac{2\pi}{8.6 \times 10^4}\right)^2 \sim 2 \times 10^5\ m^2/s^2$

SO AMPLITUDE OF TIDES IS

$$|\eta| \sim \frac{3\,G M_m h}{4\Omega^2 D^3} \sim \frac{3}{4}\frac{g}{\Omega^2}\frac{M_m}{M_0}\left(\frac{a}{D}\right)^3 \frac{h}{a}$$

~ 1.2 METER


Star Formation

The Stability of a Spherical Nebula
J. H. Jeans

PHILOSOPHICAL TRANSACTIONS.

I. *The Stability of a Spherical Nebula.*

By J. H. JEANS, *B.A., Fellow of Trinity College, and Isaac Newton Student in the University of Cambridge.*

Communicated by Professor G. H. DARWIN, *F.R.S.*

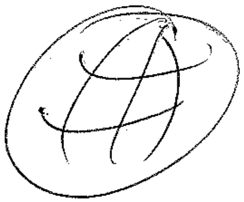
Received June 15,—Read June 20, 1901. Revised February 28, 1902.

INTRODUCTION.

§ 1. THE object of the present paper can be best explained by referring to a sentence which occurs in a paper by Professor G. H. DARWIN.* This is as follows :—

" The principal question involved in the nebular hypothesis seems to be the stability of a rotating mass of gas; but, unfortunately, this has remained up to now an untouched field of mathematical research. We can only judge of probable results from the investigations which have been made concerning the stability of a rotating mass of liquid."

Jeans Instability: Driven Sound Waves



Spherical symmetry

Uniform initial density

Force self attraction = $-\rho \nabla \Phi$

$$\nabla^2 \Phi = 4\pi G \rho$$

With ρ = constant (for some radius $r < a$)

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \rho_0$$

$$\text{So} \quad \Phi_0(r) = \begin{cases} \frac{2\pi}{3} G \rho_0 r^2 & r < a \\ \frac{2\pi}{3} G \rho_0 a^3 / r & r > a \end{cases}$$

Equilibrium forces must be balanced with increasing pressure as $r \to 0$

Linearized Self-Gravitational Response

$$\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot (\rho_0 \tilde{v}) = 0$$

$$\rho_0 \frac{\partial \tilde{v}}{\partial t} = -\nabla \tilde{p} - \rho_0 \nabla \tilde{\Phi}$$

$$\nabla \tilde{p} = c_s^2 \nabla \tilde{\rho} \quad \text{SINCE} \quad \left(\frac{p}{\rho^\gamma}\right) \sim \text{CONSTANT}$$

WITH $\nabla^2 \tilde{\Phi} = 4\pi G \tilde{\rho}$

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \nabla \cdot \tilde{v} \Rightarrow \frac{\partial^2 \tilde{\rho}}{\partial t^2} = -\rho_0 \nabla \cdot \frac{\partial \tilde{v}}{\partial t}$$

$$\nabla \cdot \frac{\partial \tilde{v}}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla^2 \rho - \nabla^2 \tilde{\Phi} \longrightarrow \frac{\partial^2 \tilde{\rho}}{\partial t^2} = c_s^2 \nabla^2 \tilde{\rho} + \rho_0 \nabla^2 \tilde{\Phi}$$

$$= c_s^2 \nabla^2 \tilde{\rho} + 4\pi G \tilde{\rho} \rho_0$$

$$\omega^2 = k^2 c_s^2 - \underbrace{4\pi G \rho_0}$$

$$k_J^2 \equiv \frac{4\pi G \rho_0}{c_s^2} \qquad = c_s^2 (k^2 - k_J^2)$$

IF $k^2 < k_J^2$, THEN SELF-ATTRACTION IS UNSTABLE!

Summary

- Linearization of dynamical equations provides a powerful methodology to understand disturbances
- Harmonic analysis (ω) and modal analysis (k) provide understanding of linear PDEs