Viscous Decay of Line Vortex

A line vortex $U_{\theta}(r) = \frac{r}{2\pi}$

Vorticity is "concentrated" along axis.

Axisymmetric $U = (0, U_{\theta}, 0) = (0, \frac{r^2}{2n}, 0)$

Navier-Stokes in cylindrical coordinates (Appendix B)

A: $- \frac{U_{\theta}^2}{\rho} = -\frac{1}{\rho} \frac{2\rho}{2n} \Rightarrow$ Pressure increases away from axis

B: $\frac{2U_{\theta}}{\rho} + (\frac{\partial}{\partial r})U_{\theta} = \frac{M}{\rho} \left( \frac{1}{\sqrt{2n}} \left( \frac{2U_{\theta}}{2n} \right) - \frac{U_{\theta}}{\rho^2} \right)$

\[
\frac{2U_{\theta}}{\rho} = \frac{M}{\rho} \left( \frac{2U_{\theta}}{2n} + \frac{1}{\sqrt{2n}} \frac{2U_{\theta}}{2n} - \frac{U_{\theta}}{\rho^2} \right)
\]

Can we convert this complicated PDE into an easier-to-solve ODE?

**Answer:** Yes! Try $U_{\theta} = F(r) \frac{r}{2\pi n} \left( \frac{1}{\sqrt{2n}} \right)$

$\mathcal{K} / \rho \in \mathbb{V}$
Similarity Solution

In some problems, like diffusion processes, the solution changes in time but as a sequence of "similar" profiles

\[ F(\eta) \sim e^{-n^2/4\gamma t} \]

In diffusion problems profile "width" scales like \( \sqrt{t} \)

So \( n^2 = \gamma t \) describes the location of "pooling"

of profiles.

\[ u_0(\eta, t) = F(\eta) \frac{n}{2\pi n} \left( \frac{n^2}{4\gamma t} \right) \]

When \( F(\eta=0) \), then \( t \to 0 \) and \( F(0) = 0 \) since flow must dissipate.

When \( F(\eta=0) \), then \( t \to 0 \) and \( F(0) = 1 \) as initial condition.

Need to know that similarity solution converts PDE to ODE.

Similitude Solution

Does similarity solution work?

\[ u_0(\eta, t) = F(\eta) \frac{n}{2\pi n} \]

\[ \frac{2u_0}{\partial t} = F' \frac{n}{2\pi n} \frac{n}{2\pi n} = -F' \frac{n^2}{4\gamma t} \frac{n}{2\pi n} \]

\[ \frac{2u_0}{\partial \eta} = F' \frac{n}{2\pi n} \frac{n}{2\pi n} - \frac{u_0}{n} \frac{2n}{2n} = \frac{n}{2\gamma t} \]

\[ \frac{\partial^2 u_0}{\partial \eta^2} = F'' \left( \frac{2n}{2\pi n} \right)^2 \frac{n}{2\pi n} \frac{2n}{2n} = -\frac{1}{2} \frac{2u_0}{\partial \eta} + \frac{u_0}{n} \]

\[ n \text{ NAVIER-STOKES:} \quad \frac{2u_0}{\partial t} = \nu \left( \frac{\partial^2 u_0}{\partial \eta^2} + \frac{1}{n} \frac{\partial u_0}{\partial \eta} - \frac{u_0}{n} \right) \]

Becomes

\[-F' \frac{n^2}{4\gamma t} \frac{n}{2\pi n} \frac{2n}{2n} = \gamma F'' \left( \frac{2n}{2\pi n} \right)^2 \frac{n}{2\pi n} = F'' \gamma \frac{n^2}{4\gamma t^2} \frac{2n}{2n} \]

\[ -F' = F'' \quad \text{YES! AN ODE...} \]

\[ \frac{2n}{2n} = -F' \implies F' = C_1 e^{-\eta} \]

\[ F(\eta) = F(0) + C_1 e^{-\eta} + C_2 \]

\[ = 1 - e^{-\eta} \quad \text{Satisfies boundaries} \]

\[ F(0) = 0 \]

\[ F(\infty) = 1 \]
**Vorticity Diffusion**

\[ u_\theta(\gamma, t) = \frac{1}{2\pi \gamma} \left( 1 - e^{-\gamma^2/4\nu t} \right) u_0 \]

Vorticity:
\[ \mathcal{V} = \nabla \times u = \frac{1}{2\pi} \left( \mathcal{V} u_0 \right)\]
\[ = \frac{1}{2\pi} \frac{2}{\gamma} \left( -e^{-\gamma^2/4\nu t} \right) = \frac{1}{2\pi} \left( \frac{1}{\gamma} \right) e^{-\gamma^2/4\nu t} \]
\[ = \frac{1}{\gamma \nu t} e^{-\gamma^2/4\nu t} \]

As used in HW#1 Problem #1

**What is a Vortex Sheet?**

Imagine a line of an "infinite" array of point/line vortices.

\[ \Gamma = \oint \mathbf{a} \cdot d\mathbf{l} = \sum \oint \mathbf{a} \cdot d\mathbf{l} = (u_1 - u_2) \, d\ell \]

with \( \mathbf{u} = (u_1(\gamma), 0, 0) \)

Names:
- Stokes: \( \frac{2u}{\nu} = \frac{\gamma u}{2\nu} \)

At \( t=0 \) \( u(y, 0) = \begin{cases} +u_0 & \text{if } y > 0 \\ -u_0 & \text{if } y < 0 \end{cases} \)

As \( t \to \infty \) \( u(y, 0) = \begin{cases} +u_0 & \text{if } y \to +\infty \\ -u_0 & \text{if } y \to -\infty \end{cases} \)

The similarity solution
\[ u(\gamma, t) = F(\gamma) u_0 \]
where \( \gamma = y / 2 \sqrt{\nu t} \)

Does this convert PDE into an ODE?

(answer: yes!)
Vortex Sheet Diffusion

\[ \vec{u} = (u_\infty(x), 0, 0) \quad \frac{\partial \vec{u}}{\partial t} = \sqrt{\frac{3}{2\tau}} \quad u(y,t) = F(\eta) u_0 \]

\[ \eta = \frac{y}{a \sqrt{\tau}} \]

\[ \frac{2U}{\partial t} = F' \frac{\partial \eta}{\partial t} = -\frac{1}{2} \frac{\partial \eta}{\partial x} F' \]

\[ \frac{2U}{\partial y} = F' \frac{\partial \eta}{\partial y} \]

\[ \frac{2U}{\partial t} = F''(\frac{\partial \eta}{\partial t})^2 = F'' \frac{1}{4\tau \epsilon} \]

- Integrating once:
  \[ \frac{df}{f'} = -2\eta \, d\eta \quad \Rightarrow \quad f' = e^{-\eta^2} \]

- Integrating again:
  \[ F(\eta) = Erf(\eta) \]

  \[ \frac{\partial}{\partial \eta} \int_{-\infty}^{\eta} d\eta \, e^{-\eta'^2} \]

\[ \Rightarrow \quad \eta = \frac{3}{\sqrt{\pi}} \int_{-\infty}^{\eta} d\eta' \, e^{-\eta'^2} \]

\[ \text{Vorticity} = \lambda = \nabla \times \vec{u} = \frac{-2U_\infty}{\partial y} = -\frac{\partial u}{\partial y} = e^{-\eta^2} \]

Error Function

\[ \text{In}[2]:= \text{Plot}[	ext{Erf}[x], \{x, 0, 5\}, \text{PlotLabel} \rightarrow \text{"Erf}[x"]], \text{PlotRange} \rightarrow \text{All}] \]

\[ \text{Out}[2]= \text{Erf}[x] \]

\[ \text{In}[3]:= \text{D}[	ext{Erf}[x], x] \]

\[ \text{Out}[3]= \frac{2 \, e^{-x^2}}{\sqrt{\pi}} \]
Analogous to Magnetic Diffusion

\[ \nabla \cdot \vec{U} = 0 \quad \nabla \times \vec{U} = \vec{0} \]
\[ \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} \]

But \( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \)
\( \vec{E} = \sigma \vec{J} \quad \sigma = \text{electrical conductivity} \)

The Navier-Stokes
\[ \frac{\partial \vec{B}}{\partial t} = -\sigma \nabla \times \vec{J} = -\sigma \nabla \times (\nabla \times \vec{B}) \]
\[ = \frac{\sigma}{\mu_0} \nabla^2 \vec{B} \]

So \( \vec{B} \) in flow
\( \frac{\sigma}{\mu_0} \sim \text{kinematic viscosity} \)
\( \vec{J} \sim \text{vorticity} \)

Viscous diffusion \( \Leftrightarrow \) Magnetic diffusion through a conductor

Stokes Flow
Steady, "Creeping", Flow Around an Object

\[ \rho \nabla \cdot \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} \]

**CASE #1: High Reynolds Number Flow**

- **Non-linear**
- **Pressure Forces Overcome Inertial Effects**

**Note:** \( \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left( \frac{1}{2} \mathbf{u}^2 \right) + \mathbf{j} \times \mathbf{u} \)

**So for Inertial Flow**

\[ \rho \nabla \left( \frac{1}{2} \mathbf{u}^2 \right) = -\nabla p \]

or \( \frac{1}{2} \mathbf{u}^2 + \rho g = \text{constant} \)

**CASE #2: Low Reynolds Number Flow**

Linear

\[ \nabla p \propto \mu \nabla^2 \mathbf{u} \]

valid only near object with strong \( \mathbf{u} \)

Stokes' Solution for Viscous Flow Around a Sphere (1851)

**Problem:** Solve \( \nabla \mathbf{p} = \mu \nabla^2 \mathbf{u} \) in spherical coordinates with no-slip boundary conditions and uniform axial flow far from sphere.

**Note:** \( \rho = \text{constant} \)

\( \nabla \cdot \mathbf{u} = 0 \)

Incompressible

\( \nabla \cdot \mathbf{j} = 0 \)

This problem has complicated algebra! (See Appendix B).

To simplify: Note

\[ \nabla^2 \mathbf{u} = \nabla \left( \nabla \cdot \mathbf{u} \right) - \mathbf{j} \times (\nabla \times \mathbf{u}) \]

\[ \nabla^2 \mathbf{j} = \nabla \left( \nabla \cdot \mathbf{j} \right) - \mathbf{j} \times (\nabla \times \mathbf{j}) \]

In spherical coordinates

\( \nabla \times (\mathbf{j} \times \mathbf{u}) \) is easier than \( \nabla^2 \mathbf{j} \)

\( \nabla \times (\mathbf{j} \times \mathbf{u}) \) is easier than \( \nabla^2 \mathbf{u} \)
Stokes Flow (1-3)

STEP #1

\[ \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} \]

Note: Neglect fluid inertia because viscous forces \( \gg \) pressure forces.

STEP #2

Use spherical coordinates.

\[ \nabla \cdot \mathbf{U} = \frac{\partial}{\partial r} (r \mathbf{U}) - \nabla \times (r \times \mathbf{U}) \]

\[ \nabla^2 \mathbf{U} = \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial \theta^2} \right) \mathbf{U} \]

STEP #3

Since flow is axi-symmetric, define a stream function:

\[ \mathbf{u} = -\nabla \times \nabla \psi \]

\[ = -\frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \nabla \psi \]

\[ = \frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \frac{2 \psi}{2 \theta} - \frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \frac{2 \psi}{2 \theta} \]

Stokes Flow (4-5)

STEP #4

Since \( \nabla \times \nabla \psi = 0 \), find a solution to stream function that satisfies:

\[ \nabla \times (\nabla \times \mathbf{U}) = 0 \]

\[ \mathbf{U} = -\frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \frac{2 \psi}{2 \theta} - \frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \frac{2 \psi}{2 \theta} \]

With some algebra...

\[ \mathbf{U} = -\frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \frac{2 \psi}{2 \theta} - \frac{\mathbf{e}_z}{\rho \sin \theta} \cdot \frac{2 \psi}{2 \theta} \]

STEP #5

Far from sphere (\( \rho \to \infty \)), we know

\[ \psi(\rho, \theta) = \frac{1}{2} \rho^2 \sin^2 \theta \]

Thus, try a solution of form:

\[ \psi(\rho, \theta) = f(\rho) \sin^2 \theta \]

where \( f(\rho \to \infty) = \frac{1}{2} \rho^2 \) and \( f(\rho \to 0) = 0 \).
Stokes' Solution (some algebra...)

\[ \mathbf{u} = -\frac{1}{\nu} \left[ \frac{1}{\sin \theta} \frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right] \]

\[ \nabla \times \mathbf{u} = \frac{1}{\sin \theta} \frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \nabla \times (\mathbf{u} \times \mathbf{u}) = \frac{1}{\sin \theta} \frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \left( \frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right) \]

\[ = 0 \]

Substitute \( \psi(\theta) = \psi(\theta) \sin \theta \) and note:

\[ \sin \frac{\theta}{2} \left( \frac{\frac{2\psi}{\cos \theta}}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right) = -\frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \]

Stokes Flow (6-7)

Step #6

After substituting...

\[ \nabla \times (\nabla \times \mathbf{u}) = \left[ \frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right]^2 \psi(\theta) = 0 \]

\[ \psi(\theta) = \frac{1}{2} \sin \theta \sin^2 \theta \left[ 1 - \frac{2\frac{\psi}{\cos \theta}}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right] \]

\[ \psi(\theta) = \frac{1}{2} \sin \theta \left[ 1 - \frac{2\frac{\psi}{\cos \theta}}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right] \]

\[ U_x(\theta) = U \cos \theta \left( 1 - \frac{2\frac{\psi}{\cos \theta}}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right) \]

\[ U_y(\theta) = -U \sin \theta \left( 1 - \frac{2\frac{\psi}{\cos \theta}}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \right) \]

Step #7

Find pressure by integrating (solving)

\[ \nabla \cdot \mathbf{u} = -\frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \]

After substituting solution for \( \mathbf{u} \), easy to show

\[ P(\theta) - P_{\infty} = -\frac{2\psi}{\frac{2\psi}{\cos \theta} + \frac{1}{\frac{2\psi}{\cos \theta}}} \]

\[ \lim_{\theta \to \pi^+} P(\theta) = \text{low pressure} \]
Stokes' Solution (some algebra...)

\[
\left[ \frac{2\gamma}{2\alpha^2} + \frac{\sin \theta}{\alpha^2} \left( \frac{1}{\sin \theta} - \frac{2}{2\alpha} \right) \right]^3 \psi(\alpha, \theta) = 0
\]

\[
\left[ \frac{2\gamma}{2\alpha^2} - \frac{z}{\alpha^2} \right]^2 \psi(\alpha, \theta) = 0
\]

\[
\left[ \frac{2\gamma}{2\alpha^2} - \frac{2}{\alpha^2} \right] - \frac{2^2}{\alpha^2} \left( \frac{z}{\alpha^2} \right) + \frac{4}{\alpha^4} \right] \psi(\alpha, \theta) = 0
\]

or

\[
f'''' - \frac{4f'''}{\alpha^2} + \frac{8f''}{\alpha^3} - \frac{8f}{\alpha^4} = 0
\]

\[f(\alpha) = \frac{a_0}{\alpha^4} + a_2 \alpha^2 + a_4 \alpha + \frac{a_6}{\alpha^3} \quad (\text{see p. 804})\]

As \( \alpha \to \infty \), 
\[f(\alpha) = \frac{1}{2} \frac{U^2}{\alpha^2} \]

As \( \alpha \to 0 \), 
\[f(\alpha) = 0 \quad : \quad \psi(\alpha, \theta) = \frac{1}{2} \frac{U^2}{\alpha^2} \sin^2 \theta \left( 1 - \frac{3a^2}{2a^4} + \frac{a^2}{2a^2} \right)\]

Stokes Flow (8-9)

**Step #8**

**FINO VISCOUS STRESS AT SURFACE OF SPHERE**

\[6n_n = 2\mu \frac{2U}{2\alpha} = 2\mu \frac{U}{\alpha} \cos \theta \left[ \frac{2a^2}{2a} - \frac{2a^3}{2a^4} \right] = 0\]

\[6n_\theta = \mu \left[ \frac{n}{2\alpha} \left( \frac{U}{2\alpha} \right) + \frac{1}{2} \frac{2U}{2\alpha} \right] = \frac{3\mu U a^3}{2a^4} \sin \theta\]

**Step #9**

**INTEGRAL TOTAL DRAG FORCE IN DIRECTION OF FLOW...**

\[\mathbf{F} = \mathbf{E} \cdot \mathbf{A} \quad \text{AT SURFACE}\]

\[\mathbf{F} \cdot \hat{\mathbf{r}} = -P \cos \theta \cos \phi + 6a \cos \theta - 6a \sin \theta\]

\[= \frac{3aU}{2a} \cos^2 \theta - P_a \cos \theta + C + \frac{3aU}{2a} \sin \theta\]

\[= \frac{3aU}{2a}\]

**TOTAL DRAG = \( 4\pi a^2 \times \frac{3aU}{2a} = 6\pi a \mu U \)**

\[(\text{WHAT IS TERMINAL VELOCITY?})\]

\[= \left( \frac{2Y}{R_0} \right) \left( \frac{1}{2} \mu U \pi a^2 \right)\]
Stokes' Law

\[ F_d = 6\pi \mu RV = \frac{24}{Re} \times \left( \frac{1}{2} \rho V^2 \pi R^2 \right) \]

\[ F_g = (\rho_p - \rho_f) g \frac{4}{3} \pi R^3, \]

\[ V_s = \frac{2(\rho_p - \rho_f)}{\mu} g R^2 \]


CGS unit for \( \mu \) is "poise" (Pl) and for \( \mu/\rho \) is "stokes" (St)

**Problem 9.1**

1. Consider the laminar flow of a fluid layer falling down a plane inclined at an angle \( \theta \) with the horizontal. If \( h \) is the thickness of the layer in the fully developed stage, show that the velocity distribution is

\[ u = \frac{g \sin \theta}{2v} (h^2 - y^2), \]

where the \( x \)-axis points along the free surface, and the \( y \)-axis points toward the plane. Show that the volume flow rate per unit width is

\[ Q = \frac{gh^3 \sin \theta}{3v}, \]

and the frictional stress on the wall is

\[ \tau_0 = \rho gh \sin \theta. \]
Problem 9.1

\[ \nabla \cdot \mathbf{u} = -\dot{\gamma} + \nu \nabla^2 \mathbf{u} \]

\[ \xi \text{- component} \]

\[ \dot{\gamma} = g \sin \theta + \frac{\nu}{2} \nabla^2 \mathbf{u} \]

Solve for \( \mathbf{u}_x \):

\[ \mathbf{u}_x(t) = c_1(t) + c_2 \frac{g \sin \theta}{2\nu} \]

Boundary conditions:

At \( r = 0 \), \( \mathbf{u}_x(r) = 0 \) \hspace{1cm} c_1 = 0

At \( r = h \), no stress at free surface, so \( \frac{\partial \mathbf{u}_x}{\partial r} = 0 \) \hspace{1cm} c_2 = \frac{g h}{2\nu} \sin \theta

\[ \mathbf{u}_x(r) = \frac{g}{\nu} \frac{r h}{2} \sin \theta \left( 2 - \frac{r}{h} \right) \]

Flow rate \( Q = \frac{g}{\nu} \frac{h^3}{2} \sin \theta \int_0^h dy \left( 2 - \frac{r}{h} \right) = \frac{gh^3}{3} \sin \theta \)

Shear stress \( \mu \frac{2u}{A} = \mu \frac{g}{\nu} h \left( 1 - \frac{r}{h} \right) = g \rho h \sin \theta \left( 1 - \frac{r}{h} \right) \)

HW #3

- Viscous decay of a line vortex
- Stream lines and vortex lines
- Leap-frog vortex rings
- Viscous relaxation towards steady flow
Summary

- Viscosity causes vorticity diffusion and decreases velocity gradients.

- For strong viscosity, viscous forces balance pressure forces. Drag coefficient scales inversely with Reynolds number as expected.