1. Dimensional analysis (again)
2. Laminar flow (when viscosity exceeds advection)
3. Examples: Poiseuille's steady flow through a pipe

**Dimensional Variables**

**Continuity:** \( \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = -\rho \nabla \cdot \vec{u} \)

**Navier-Stokes:** \( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = \nabla \psi + \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{u} \)

Flow problems are characterized by:

- **Length Scale** = \( L^* \)
- **Flow Speed** = \( U^* \)

This sets a characteristic time \( t^* = L^*/U^* \).

**Dimensionless Variables:**

\( t' = t/t^* \quad \vec{A} = L^* \vec{A} \quad \vec{u} = \vec{u}/U^* \)
Dimensionless Equations

Continuity
\[ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \]

Navier-Stokes
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\mathbf{g} L^*}{U_0^2} - \frac{1}{\rho U_*^2} \nabla \rho + \left( \frac{\rho}{\rho} \right) \mathbf{u} \cdot \nabla \mathbf{u} \]

If we define a characteristic pressure as twice the dynamic pressure,
\[ \rho^* = \rho U_*^2 \]

Then
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\mathbf{g} L^*}{U_0^2} - \nabla \rho' + \left( \frac{\rho}{\rho_0} \right) \mathbf{u} \cdot \nabla \mathbf{u} \]

Froude Number
Reynolds Number

Incompressible NS

\[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} \]

\[ 0 \approx -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} \]

Steady Laminar Flow!
Laminar Flow (Ch. 9)

- Two simple examples:
  - Couette Flow (moving top plate)
  - Poiseuille Flow (pressure-driven)

⇒ Steady with viscosity

Maurice Couette

Figure 1. Maurice Couette (1858-1943)

The photograph represents Maurice Couette carrying out a cathetometer observation at the Faculty of Angers.

According to his family, Maurice Couette could not conceive of a measurement being made without taking all the necessary precautions beforehand. He would insist on the experimenter's work top being well organised, with the instruments and the experimenter himself being properly installed.
Jean Louis Marie Poiseuille

In 1838 he experimentally derived and in 1840 and 1846 formulated and published Poiseuille's law for laminar stationary flow of an incompressible uniform viscous liquid (so-called Newtonian fluid) through a cylindrical tube with constant circular cross-section.

He applied his formula to blood flow in capillaries and veins, to air flow in lung alveoli, for the flow through a drinking straw or through a hypodermic needle. The unit of viscosity, Poise was named after him. (Ch. 17 !)
Poiseuille Flow

Fig. 9.14. Establishment of parabolic velocity profile in Poiseuille tube flow (redrawn from Goldsmith and Turitto\textsuperscript{386}).

Steady Laminar Flow

\[
\frac{\partial u}{\partial x} = \frac{1}{\rho} \frac{d}{d\gamma} \frac{d}{d\gamma} (\rho \frac{d}{d\gamma} u) = \frac{2\mu}{\rho d^2} \frac{d}{d\gamma} u
\]

\[
\frac{d}{d\gamma} \frac{d}{d\gamma} u = 0
\]

\[
\frac{d}{d\gamma} u = 2 \mu \frac{d^2}{d\gamma^2} u
\]

\[
\begin{align*}
\text{INTEGRATE } & \text{-component over } \gamma \ldots \\
\frac{d}{d\gamma} u &= \frac{2\mu}{\rho d^2} \frac{d}{d\gamma} u + C_1 \\
C_1 &= \text{constant}
\end{align*}
\]

\[
\begin{align*}
\text{AGAIN} & \ldots \\
\frac{d}{d\gamma} u &= \frac{2\mu}{\rho d^2} \frac{d}{d\gamma} u + C_1 + C_2 \\
C_2 &= \text{constant}
\end{align*}
\]

\[
\text{MATCH BOUNDARY CONDITIONS: } u(0) = 0 \quad u(\gamma = 2b) = u_0
\]

\[
\begin{align*}
u(0) &= 0 \Rightarrow C_2 = 0 \\
u(2b) &= u_0 \Rightarrow C_1 = \frac{1}{2\mu} \left(2b\right)^2 \frac{d}{d\gamma} u - \frac{2\mu}{d^2} \frac{d}{d\gamma} u_0
\end{align*}
\]

\[
\begin{align*}
u(\gamma) &= \frac{1}{2\mu} \frac{d}{d\gamma} \left[2b \frac{d}{d\gamma} u - 2\gamma \frac{d}{d\gamma} u + u_0 (2b)\right]
\end{align*}
\]
**Couette Flow**

Let $\frac{\partial U}{\partial x} = 0$

$U(y) = U_0 \frac{y}{b}$

$\tau_{ij} = -P \delta_{ij} + 2\mu \varepsilon_{ij}$

No net force or fluid elements

Shear stress:

$\tau_{xy} = \frac{\mu}{2} \left( \frac{2y}{b} + \frac{2y}{b} \right)$

Uniform stress:

$\tau_{xy} = \frac{\mu}{2b} (\rho U^2)$

$\tau_{xy} = \frac{1}{\text{Re}} (\rho U^2) \propto \frac{1}{\text{Re}}$

**Poiseuille Flow**

Let $U_0 \rightarrow 0$

High pressure $\rightarrow$ $\frac{1}{2} \frac{2\rho}{2x} U(x)$

Low pressure $\rightarrow$ $U(y) = \frac{1}{2} \frac{2\rho}{2x} y(y-b)$

Shear stress:

$\tau_{xy} = \frac{\mu}{2} \frac{2y}{b} = \frac{1}{2} \frac{2\rho}{2x} \left[ (y-b) + y \right]$  

$\tau_{xy} = \frac{2\rho}{2x} (y-b)$  

Independent of Re!

Net viscous force on fluid element

$\tau_{xy} \rightarrow$ Net viscous force on fluid element
Steady Laminar Flow in a Pipe

**Navier-Stokes:**
\[ \rho \vec{u} \cdot \nabla \vec{u} = - \frac{\partial p}{\partial x} + \mu \nabla^2 \vec{u} \]

\[ \vec{u} = (u, v, w) \quad U_e(\eta) \]

Use Appendix B, Cylindrical Coordinates

\[ \hat{\rho} : \quad \frac{\partial U_e}{\partial \eta} = - \frac{2p}{3\eta} + \lambda \nabla^2 U_e \]

\[ \hat{\theta} : \quad \frac{\partial U_e}{\partial \theta} = - \frac{1}{r} \frac{2p}{2\eta} + \lambda \nabla^2 U_e \]

Pressure is independent of \((\eta, \theta)\)

\[ \eta \left( \frac{2U_e}{2\eta} \right) = \frac{1}{\lambda} \frac{2p}{2\eta} \]

Integrating once:

\[ \frac{2U_e}{2\eta} = \frac{n^2}{2\mu} \frac{2p}{2\eta} + C_1 \]

Integrating twice:

\[ U_e = \frac{n^2}{2\mu} \frac{2p}{2\eta} + C_1 \eta + C_2 \]

This must vanish \(\text{why?}\) so \(C_1 = 0\)

Steady Flow (Cont.)

Boundary Conditions:

\[ U_e(\eta=a) = 0 \Rightarrow \text{No slip condition at pipe walls} \]

\[ U_e(\eta=0) \text{ is well defined} \]

So

\[ U_e(\eta) = -\frac{1}{4\mu} \frac{2p}{2\eta} (a^2 - \eta^2) \]

How much flow?

\[ Q = \int_0^a 2\pi \eta \eta \eta U_e(\eta) = -\frac{\pi}{4\mu} \frac{2p}{2\eta} \int_0^a \eta \eta \eta (a^2 - \eta^2) \]

Poiseuille's Law (Lambert's Law) for Pressure Drop across Pipe:

\[ \text{(Pressure Drop)} = \text{(Length)} \times Q \left( \frac{5\mu}{\pi \eta a} \right) \]

Two Tubes with Equal Conductance

\[ Q = 2 \]

\[ 8 \times L \to \]
Shear Stress and Force Balance

\[ \text{STRESS} = \tau_{\text{en}} = 2\mu \varepsilon_{\text{en}} \quad \text{where} \quad \varepsilon_{\text{en}} = \frac{1}{2} \frac{2u}{\Delta y} = \frac{n}{\pi D^2} \]

\[ \tau_{\text{en}} = \frac{n}{4} \frac{2\rho}{D^2} \]

Viscous Force = \( \nabla \cdot \tau = \frac{1}{\eta} \frac{2}{\Delta y} (\tau_{\text{en}}) = \mu \nabla^2 U = \frac{2\rho}{2\pi} \]

So viscous force is uniform and equal and opposite to pressure force.

Uniform steady flow occurs with a balance between pressure and viscous forces.

Time-Dependance:

Flow Relaxation

\[ t = 0 \quad \rightarrow \quad t > 0 \]

\[ U(y) = 0 \quad \rightarrow \quad U(y) \neq 0 \]

\[ \frac{2U}{2t} = \frac{M}{P} \frac{2^2 U}{2y^2} \quad \text{Boundary} \quad \begin{cases} U(0) = 0 \quad t > 0 \\ u(2b) = U_0 \quad t > 0 \end{cases} \]

With initial condition \( U(y) = 0 \) at \( t = 0 \)

Final condition \( U(y) = \frac{u_0 y}{2b} \) as \( t \to \infty \)
Fourier's Method

\[ U(y, t) = F(y) \frac{\partial^2 U}{\partial t^2} \]

\[ \frac{\partial^2 F}{\partial y^2} (16) \]

On \[ \frac{1}{2} \frac{2^6}{6} \frac{\partial^6}{\partial t^6} = \left( \frac{\partial^2}{\partial y^2} \right) \gamma \]

Then \[ \gamma(6) = 6. \quad \gamma = \frac{\partial^2 F}{\partial y^2} = -\gamma F \]

**Fourier Moore Expansion**

\[ U(y, t) = \frac{U_0}{2b} + \sum_{m=1}^{\infty} f_m \sin(\pi m \frac{y}{2b}) \]

\[ U(y, 0^+) = 0 \]

\[ U(y, t-\infty) = U_0 \left( \frac{y}{2b} \right) \]

\[ \sum_{m=1}^{\infty} f_m \sin(\pi m \frac{y}{2b}) \]

Fourier modes must **decay as** \( t \to \infty \)

**Fourier's Solution**

\[ U(y, t) = U_0 \left( \frac{y}{2b} \right) + \sum_{m=1}^{\infty} f_m e^{-\delta_m t} \sin(\pi m \frac{y}{2b}) \]

\[ \delta_m = \left( \frac{\pi m}{2b} \right)^2 = \frac{1}{R_0} \left( \frac{\pi^2 \alpha^2}{2b} \right) U_0 \]

Fourier modes with shorter wavelengths **decay to zero much more quickly than long wavelengths!!**

To find \( f_m \), we use initial condition \( U(y, 0) = 0 \)

\[ -U_0 \frac{\partial U_0}{\partial b} = \sum_{m=1}^{\infty} f_m \sin(\pi m \frac{y}{2b}) \]

\[ -U_0 \int_0^{2b} \sin(\pi m \frac{y}{2b}) dy = \sum_{m=1}^{\infty} f_m \int_0^{2b} \sin(\pi m \frac{y}{2b}) \sin(\pi m \frac{y}{2b}) dy \]

\[ \frac{U_0 (2b)^2 (-1)^m}{\pi m} = f_m \delta_{mm} \]

(1768-1830) Fourier's Method
Fourier’s Solution


\[ U(\gamma, t) = U_0 \left( \frac{\gamma}{\delta_b} \right) + \sum_{m} \frac{2 U_0 (-1)^m}{m \pi} \sin \left( \pi \frac{m}{\delta_b} \right) e^{-\gamma_m t} \]

\[ \gamma_m = \frac{1}{R_e} \left( \frac{\pi^2 m^2}{\delta_b} \right) U_0 \]

Section 9.7 gives other solution methods:

- Solution in terms of similarity variable
  \[ \eta = \frac{\gamma}{\partial U_0 / \partial t} \]
- Laplace Transform

Solution with 10 Fourier Modes

\[ U(y) \text{ at } 10^{-5}, 10^{-3}, 10^{-2}, 10^{-1}, 1 \]
Solution with 20 Fourier Modes

Solution with 50 Fourier Modes
Summary

• When the Reynolds number is not too large, and advective term is much less than viscosity, then steady flow is laminar.

• Some relatively simple problems can be solved analytically to guide our understanding of low Re viscous flow.