APPH 4200
Physics of Fluids

Laminar Flow (Ch. 9)

1. Dimensional analysis (again)
2. Laminar flow (when viscosity exceeds advection)
3. Examples: Poiseuille's steady flow through a pipe

Dimensional Variables

CONTINUITY: $\frac{\partial \rho}{\partial t} + \bar{u} \cdot \bar{\nabla} \rho = -\rho \nabla \cdot \bar{u}$

NAVIER-STOKES: $\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \bar{\nabla}) \bar{u} = \bar{g} - \frac{1}{\rho} \bar{\nabla} p + \frac{\mu}{\rho} \nabla^2 \bar{u}$

FLOW PROBLEMS ARE CHARACTERIZED BY

<u>LENGTH SCALE</u> $= L^*$

<u>FLOW SPEED</u> $= u^*$

THIS SETS A CHARACTERISTIC TIME $t^* = L^*/u^*$.

<u>DIMENSIONLESS VARIABLES</u>:

$t' = t/t^*$ $\quad \bar{\nabla}' = L^* \nabla$ $\quad \bar{u}' = \bar{u}/u^*$

Dimensionless Equations

CONTINUITY

$$\frac{\partial \rho}{\partial t'} + \bar{u}' \cdot \bar{\nabla}' \rho = -\rho \, \bar{\nabla}' \cdot \bar{u}'$$

NAVIER-STOKES

$$\frac{\partial \bar{u}'}{\partial t'} + \bar{u}' \cdot \bar{\nabla}' \bar{u}' = \frac{\bar{g} L^*}{U^{*2}} - \frac{1}{\rho U^{*2}} \nabla' p$$

$$+ \left(\frac{\mu}{\rho}\right) \frac{1}{L^* U^*} \nabla'^2 \bar{u}'$$

IF WE DEFINE A CHARACTERISTIC PRESSURE AS TWICE THE DYNAMIC PRESSURE

$$p^* = \rho U^{*2} \qquad p' = p^* p$$

THEN

$$\frac{\partial \bar{u}'}{\partial t} + \bar{u}' \cdot \bar{\nabla}' \bar{u}' = \underbrace{\frac{\bar{g} L^*}{U^{*2}}}_{F_R^{-2}} - \nabla' p' + \underbrace{\left(\frac{\mu}{\rho L^* U^*}\right)}_{\frac{1}{Re}} \bar{\nabla}'^2 \bar{u}'$$

F_R: FROUDE NUMBER

Re: REYNOLDS NUMBER

Incompressible NS

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 U$$

$$0 \approx -\nabla P + \frac{1}{Re} \nabla^2 U$$

Steady Laminar Flow!

Laminar Flow (Ch. 9)

- Two simple examples:
  - Couette Flow (moving top plate)
  - Poiseuille Flow (pressure-driven)

➡ Steady with viscosity

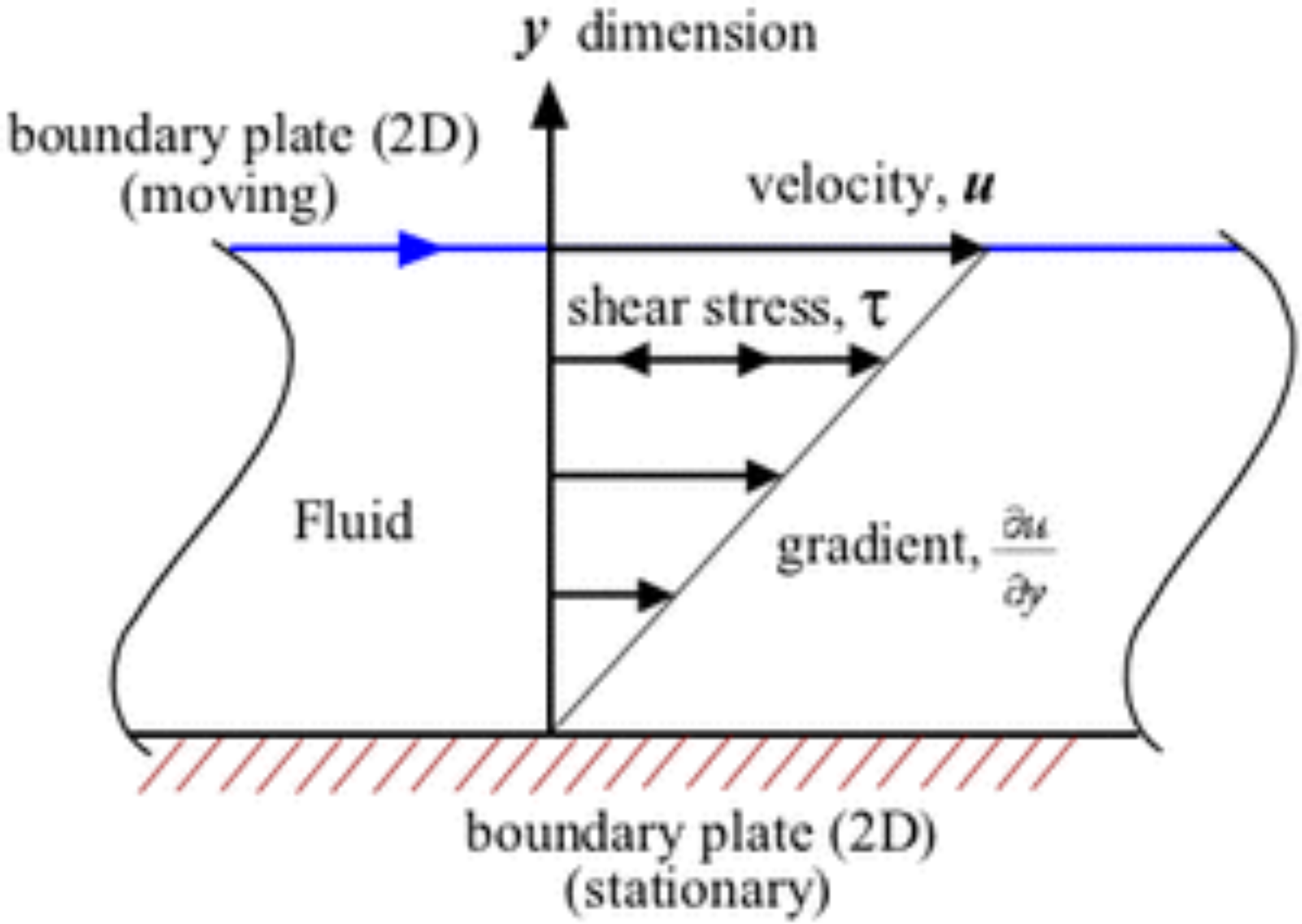
Maurice Couette

Figure 1. Maurice Couette (1858-1943)

The photograph represents Maurice Couette carrying out a cathetometer observation at the Faculty of Angers.

According to his family, Maurice Couette could not conceive of a measurement being made without taking all the necessary precautions beforehand. He would insist on the experimenter's work top being well organised, with the instruments and the experimenter himself being properly installed.

Couette Flow



Jean Louis Marie Poiseuille

In 1838 he experimentally derived and in 1840 and 1846 formulated and published Poiseuille's law for laminar stationary flow of an incompressible uniform viscous liquid (so-called Newtonian fluid) through a cylindrical tube with constant circular cross-section.

He applied his formula to blood flow in capillaries and veins, to air flow in lung alveoli, for the flow through a drinking straw or through a hypodermic needle. The unit of viscosity, Poise was named after him. (Ch. 17 !)

(1799-1869)

Poiseuille Flow

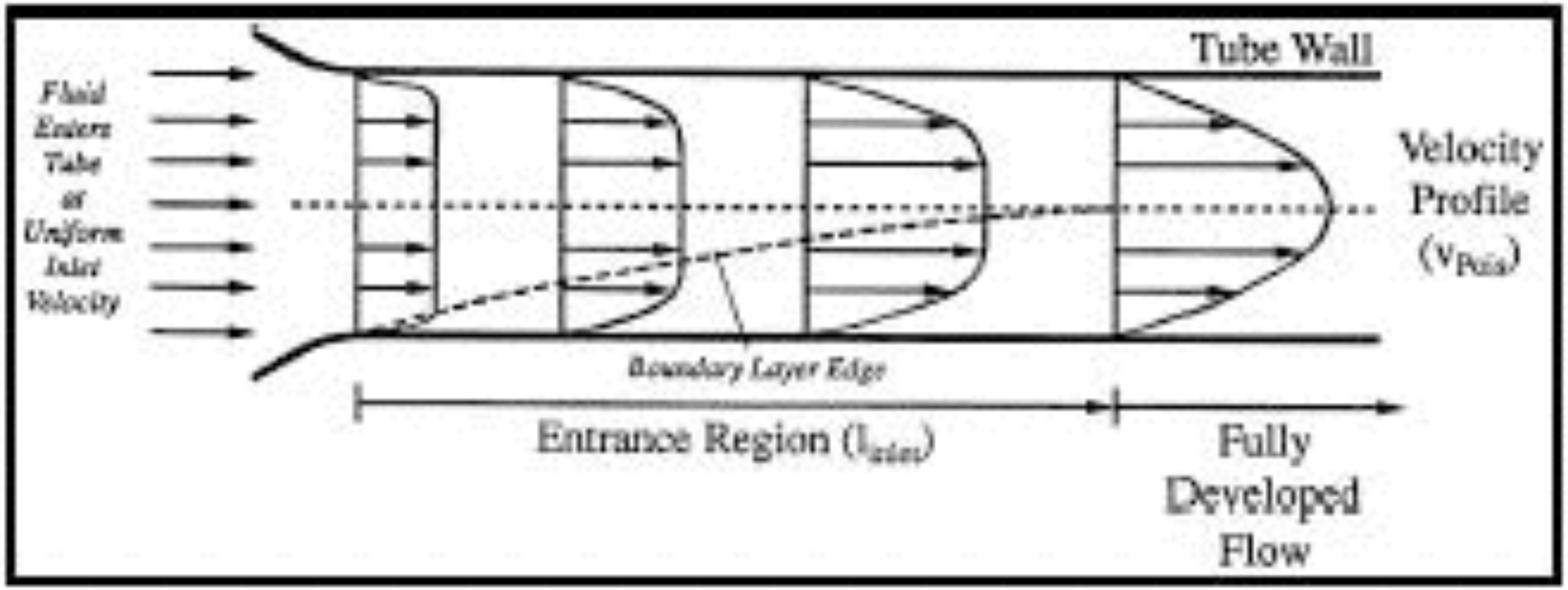
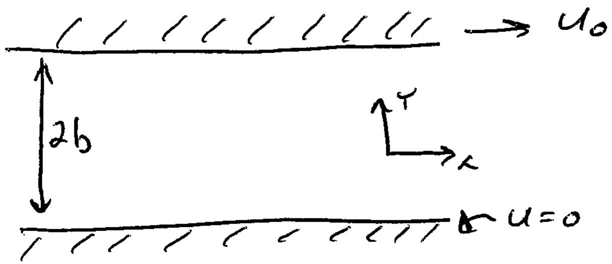


Fig. 9.14. Establishment of parabolic velocity profile in Poiseuille tube flow (redrawn from Goldsmith and Turitto[386]).

Steady Laminar Flow



$$\bar{U}\cdot\bar{\nabla}\bar{U} = -\frac{1}{\rho}\bar{\nabla}P + \frac{\mu}{\rho}\nabla^2\bar{U}$$

$$\bar{U} = \hat{x}\, U(y)$$

$\hat{y}$: $$\frac{\partial P}{\partial y} = 0$$

$\hat{x}$: $$\frac{\partial P}{\partial x} = \mu\nabla^2 U = \mu\frac{\partial^2 U}{\partial y^2}$$

INTEGRATE $\hat{x}$-COMPONENT OVER y.....

$$y\frac{\partial P}{\partial x} = \mu\frac{\partial U}{\partial y} + C_1 \qquad C_1 = \text{CONSTANT}$$

AGAIN

$$\frac{1}{2}y^2\frac{\partial P}{\partial x} = \mu U + C_1 y + C_2 \qquad C_2 = \text{CONSTANT}$$

MATCH BOUNDARY CONDITIONS: $U(0) = 0 \qquad U(y=2b) = U_0$

$$U(0) = 0 \Rightarrow C_2 = 0$$

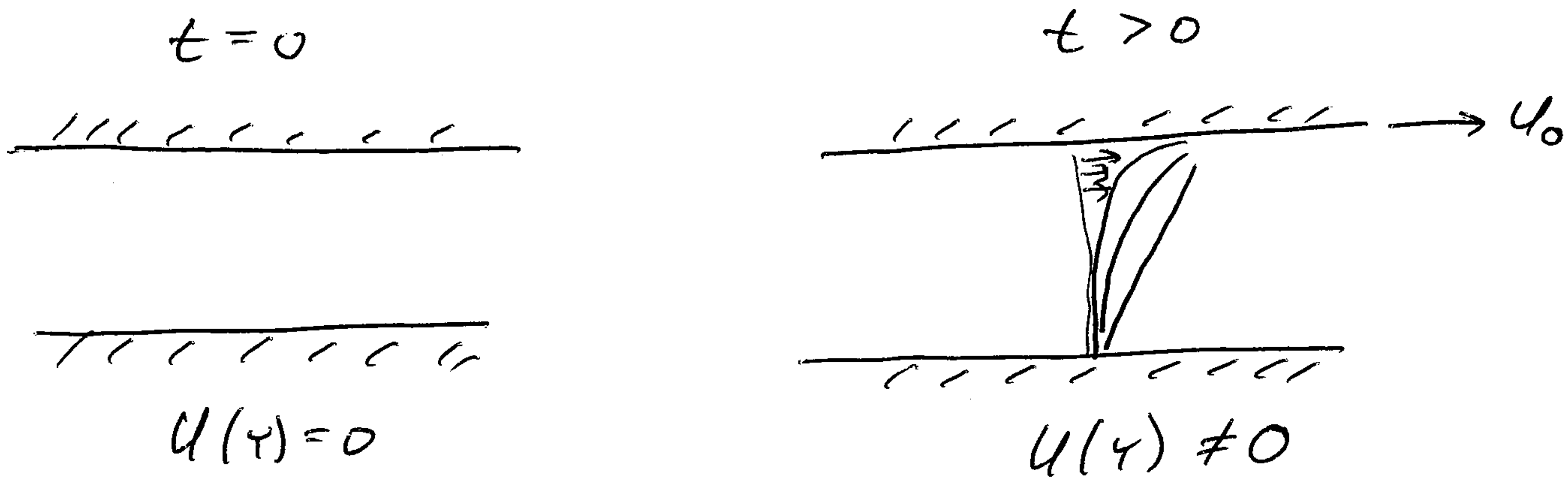
$$U(2b) = U_0 = \frac{1}{2\mu}(2b)^2\frac{\partial P}{\partial x} - \frac{C_1 2b}{\mu} \Rightarrow C_1 = b\frac{\partial P}{\partial x} - \frac{U_0\mu}{2b}$$

$$\boxed{U(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x}\, y(y-2b) + U_0\left(\frac{y}{2b}\right)}$$

POISEUILLE TERM — COUETTE TERM

Time-Dependance:

Flow Relaxation



$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Boundary
$$\begin{cases} u(0) = 0 & t > 0 \\ u(2b) = u_0 & t > 0 \end{cases}$$

With initial condition $u(y) = 0$ at $\underline{t = 0}$

Final condition $u(y) = \frac{u_0 y}{2b}$ as $\underline{t \to \infty}$

Fourier's Method

$$u(y,t) = F(y)G(t)$$

$$\frac{\partial u}{\partial t} = \left(\frac{\mu}{\rho}\right)\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial}{\partial t}(FG) = \frac{\mu}{\rho}\frac{\partial^2}{\partial y^2}(FG)$$

or $$\frac{1}{G}\frac{\partial G}{\partial t} = \left(\frac{\mu}{\rho}\right)\frac{1}{F}\frac{\partial^2 F}{\partial y^2} = -\gamma$$

THEN $G(t) = G_0 e^{-\gamma t}$ AND $\frac{\mu}{\rho}\frac{\partial^2 F}{\partial y^2} = -\gamma F$

(1768-1830)

FOURIER MODE EXPANSION

$$u(y,t) = \frac{u_0 y}{2b} + \sum_m f_m \sin\left(\pi m \frac{y}{2b}\right)$$

$$u(y,0^+) = 0$$

$$u(y,t\to\infty) = u_0\left(\frac{y}{2b}\right)$$

$\therefore$ FOURIER MODES MUST DECAY AS $t\to\infty$

Fourier's Solution

$$u(y,t) = u_0\left(\frac{y}{2b}\right) + \sum_m f_m e^{-\gamma_m t} \sin\left(\pi m \frac{y}{2b}\right)$$

$$R_e \equiv \frac{\rho 2b u_0}{\mu}$$

$$\gamma_m = \left(\frac{\mu}{\rho}\right)\left(\frac{\pi m}{2b}\right)^2 = \frac{1}{R_e}\left(\frac{\pi^2 m^2}{2b}\right) u_0$$

FOURIER MODES WITH "SHORT WAVELENGTHS" DECAY TO ZERO MUCH MORE QUICKLY THAN LONG WAVELENGTHS!!

TO FIND f_m, WE USE INITIAL CONDITION $u(y,0) = 0$

$$-u_0 \frac{y}{2b} = \sum_m f_m \sin\left(\pi m \frac{y}{2b}\right)$$

$$-\frac{u_0}{2b}\int_0^{2b} dy\, y \sin\left(\pi m \frac{y}{2b}\right) = \sum_m f_n \int_0^{2\pi} dy \sin\left(\pi n \frac{y}{2b}\right)\sin\left(\pi m \frac{y}{2b}\right)$$

$$\frac{u_0}{2b}\,\frac{(2b)^2(-1)^n}{n\pi} = f_m\, b\, \delta_{nm}$$

Fourier's Solution

THEREFORE

$$u(y,t) = u_0\left(\frac{y}{2b}\right) + \sum_m \frac{2u_0(-1)^m}{m\pi}\sin\left(\pi m\frac{y}{2b}\right)e^{-\gamma_m t}$$

$$\gamma_m = \frac{1}{Re}\left(\frac{\pi^2 m^2}{2b}\right)u_0$$

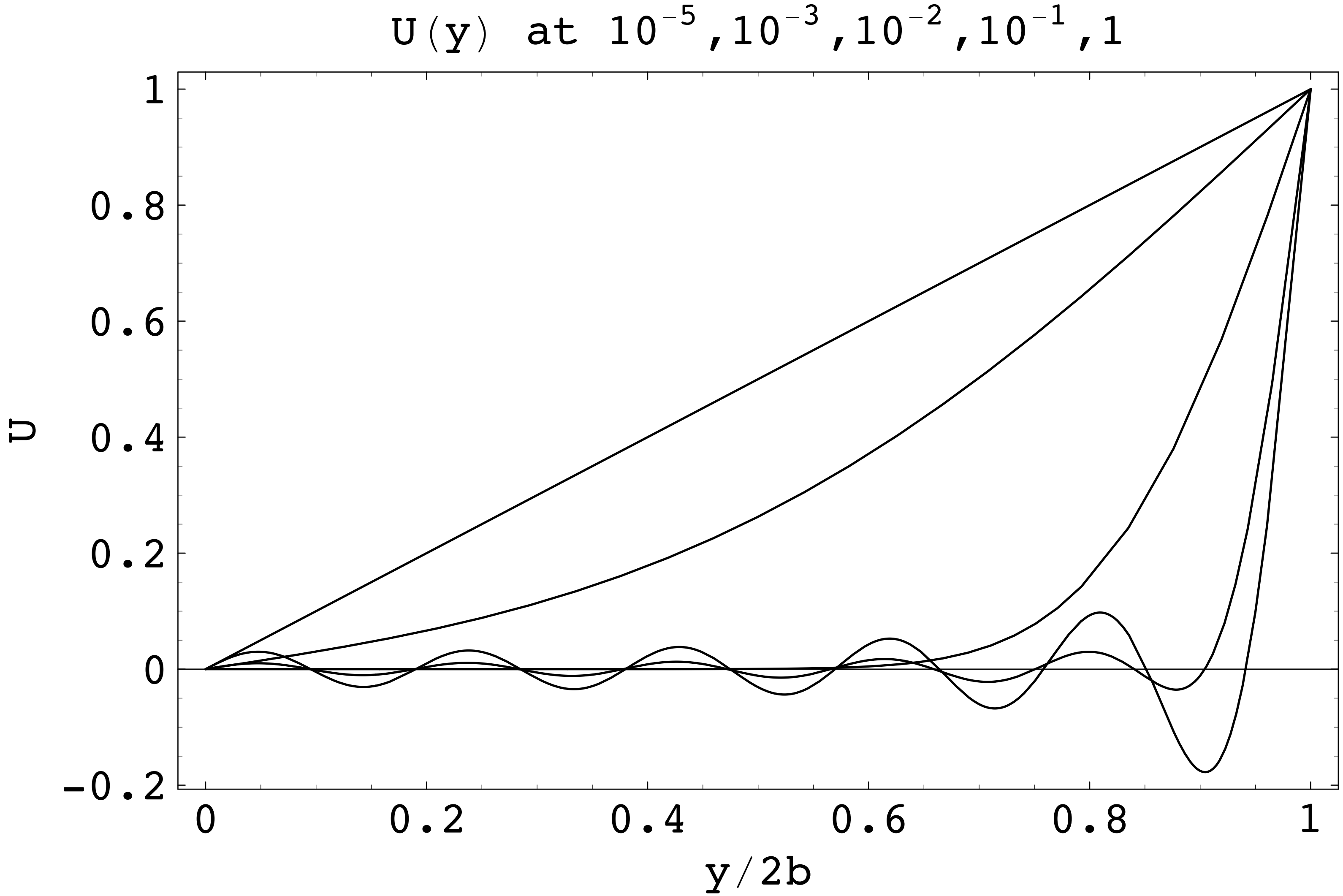
SECTION 9.7 GIVES OTHER SOLUTION METHODS:

- SOLUTION IN TERMS OF "SIMILARITY" VARIABLE

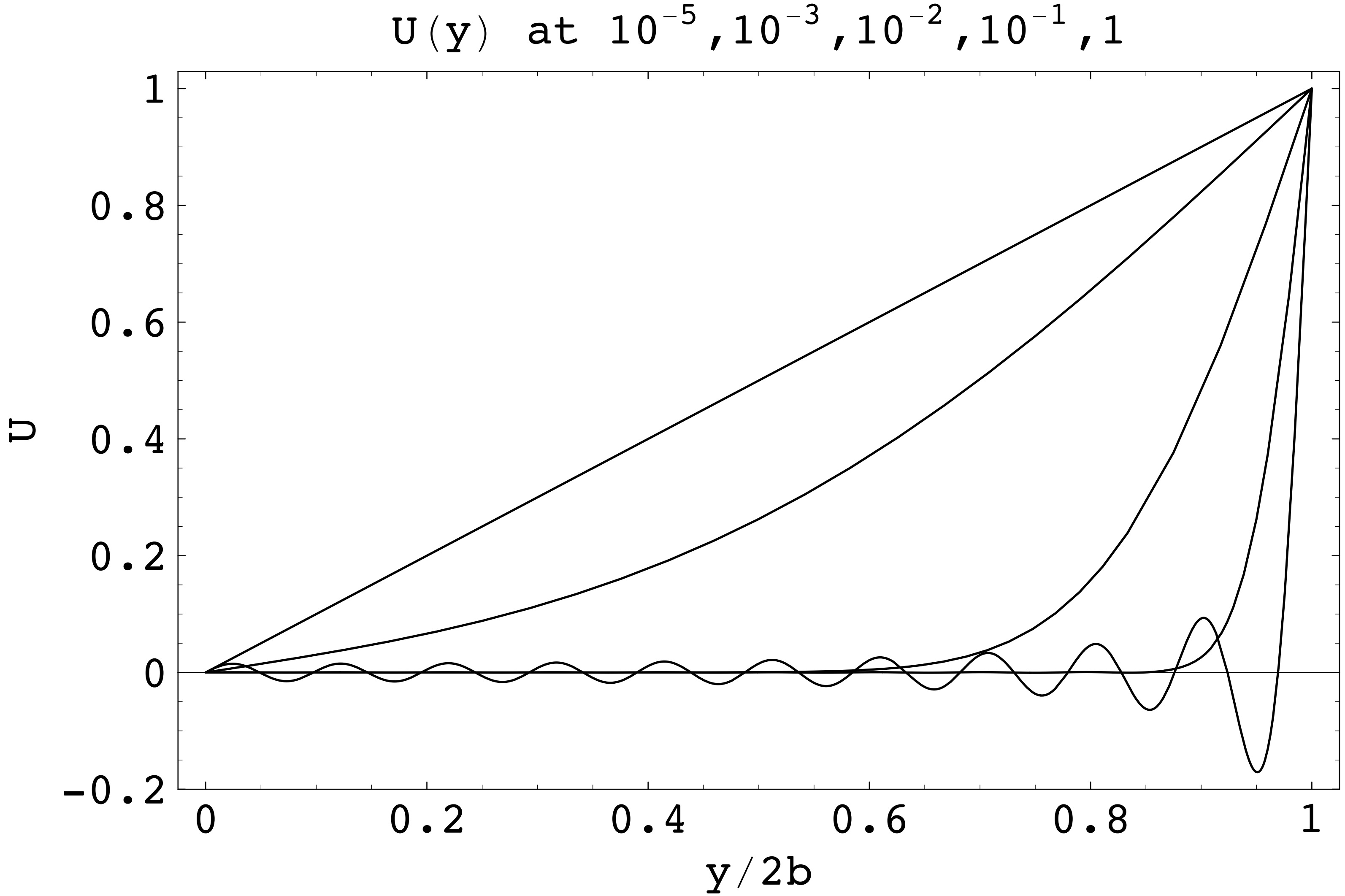
$$\eta \equiv \frac{y}{2\sqrt{\frac{\mu}{\rho}t}} \qquad U(y,t) = U_0 F(\eta)$$

- LAPLACE TRANSFORM

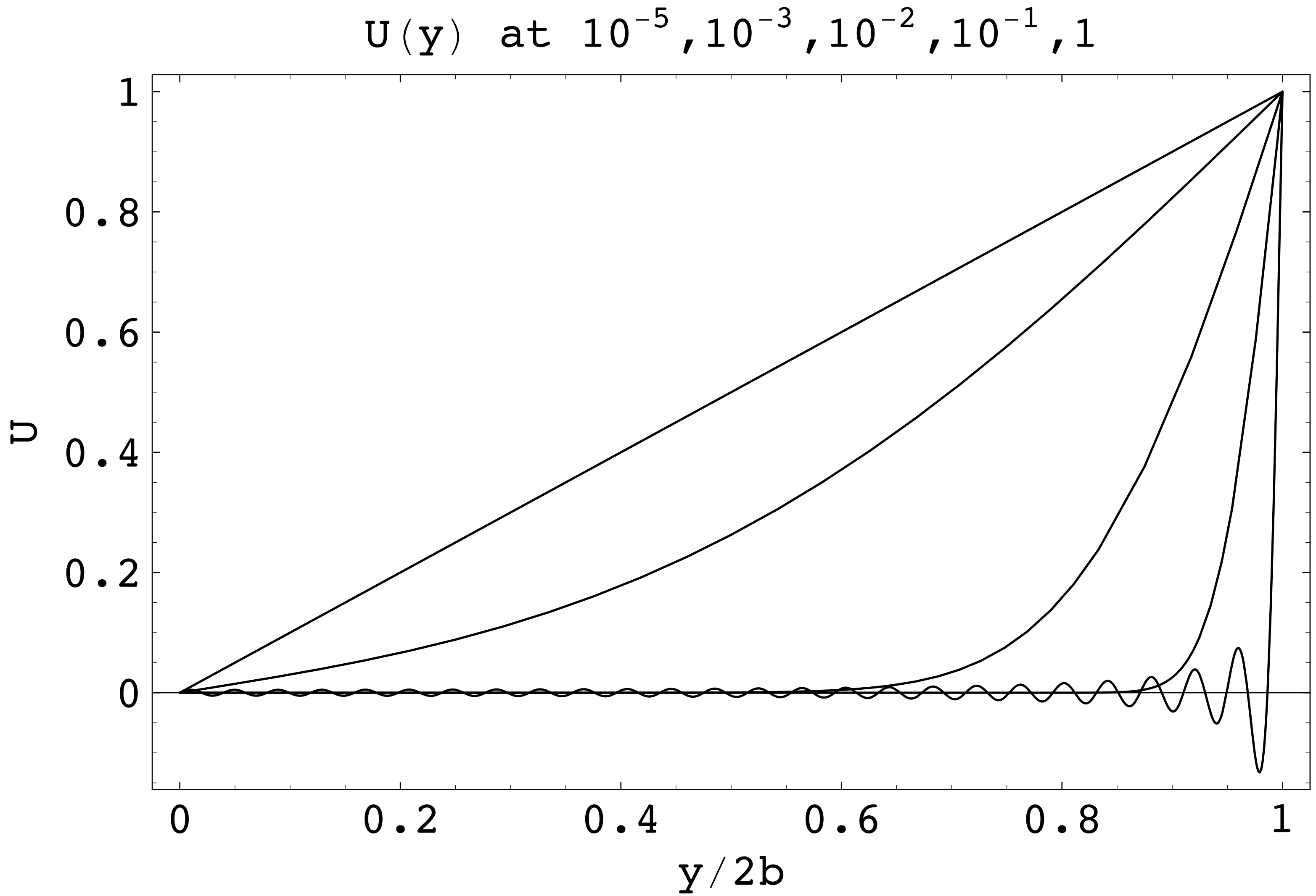
Solution with 10 Fourier Modes



Solution with 20 Fourier Modes



Solution with 50 Fourier Modes



Summary

- When the Reynolds number is not too large, and advective term is much less than viscosity, then steady flow is laminar.
- Some relatively simple problems can be solved analytically to guide our understanding of low Re viscous flow.