APPH 4200
Physics of Fluids

Similarity (Ch. 8)

1. Dimensional analysis

"Dynamic Similarity"

(or finding the key dimensionless parameters)

- Wind tunnels (powerful method in experimental fluid mechanics!)
- Physical insights (what governs dynamics and the solutions to equations)
- Convenient; significantly helps validation of models; broadens impact; defines general properties; ...

Dimensional Variables

CONTINUITY: $\frac{\partial \rho}{\partial t} + \bar{u}\cdot\bar{\nabla}\rho = -\rho\,\nabla\cdot\bar{u}$

NAVIER-STOKES: $\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u} = \bar{g} - \frac{1}{\rho}\bar{\nabla}p + \frac{\mu}{\rho}\nabla^2\bar{u}$

FLOW PROBLEMS ARE CHARACTERIZED BY

<u>LENGTH SCALE</u> $= L^*$

<u>FLOW SPEED</u> $= u^*$

THIS SETS A CHARACTERISTIC TIME $t^* = L^*/u^*$.

<u>DIMENSIONLESS VARIABLES:</u>

$t' = t/t^*$ $\quad \bar{\nabla}' = L^*\nabla$ $\quad \bar{u}' = \bar{u}/u^*$

Dimensionless Equations

CONTINUITY

$$\frac{\partial \rho}{\partial t'} + \bar{u}' \cdot \bar{\nabla}' \rho = -\rho \, \bar{\nabla}' \cdot \bar{u}'$$

NAVIER-STOKES

$$\frac{\partial \bar{u}'}{\partial t'} + \bar{u}' \cdot \bar{\nabla}' \bar{u}' = \frac{\bar{g} L^*}{U^{*2}} - \frac{1}{\rho U^{*2}} \nabla' p$$

$$+ \left(\frac{\mu}{\rho}\right) \frac{1}{L^* U^*} \nabla'^2 \bar{u}'$$

IF WE DEFINE A CHARACTERISTIC PRESSURE AS TWICE THE DYNAMIC PRESSURE

$$p^* \equiv \rho U^{*2} \qquad p' = p^* p$$

THEN

$$\frac{\partial \bar{u}'}{\partial t} + \bar{u}' \cdot \bar{\nabla}' \bar{u}' = \underbrace{\frac{\bar{g} L^*}{U^{*2}}}_{F_R^{-2}} - \nabla' p' + \underbrace{\left(\frac{\mu}{\rho L^* U^*}\right)}_{\frac{1}{R_e}} \bar{\nabla}'^2 \bar{u}'$$

$\boxed{\text{FROUDE NUMBER}}$ $\boxed{\text{REYNOLDS NUMBER}}$

Physical Similarity

IF KEY DIMENSIONLESS PARAMETERS ARE EQUAL, THEN FLOW PATTERNS AND BEHAVIORS ARE SIMILIAR. VERY USEFUL!

FROUDE NUMBER $= F_R \equiv \dfrac{U^*}{\sqrt{gL^*}}$

$= \dfrac{\left(\text{CHARACTERISTIC FLOW SPEED}\right)}{\left(\text{CHARACTERISTIC GRAVITY WAVE SPEED}\right)}$

NOTE: SPEED OF A WATER SURFACE WAVE IS

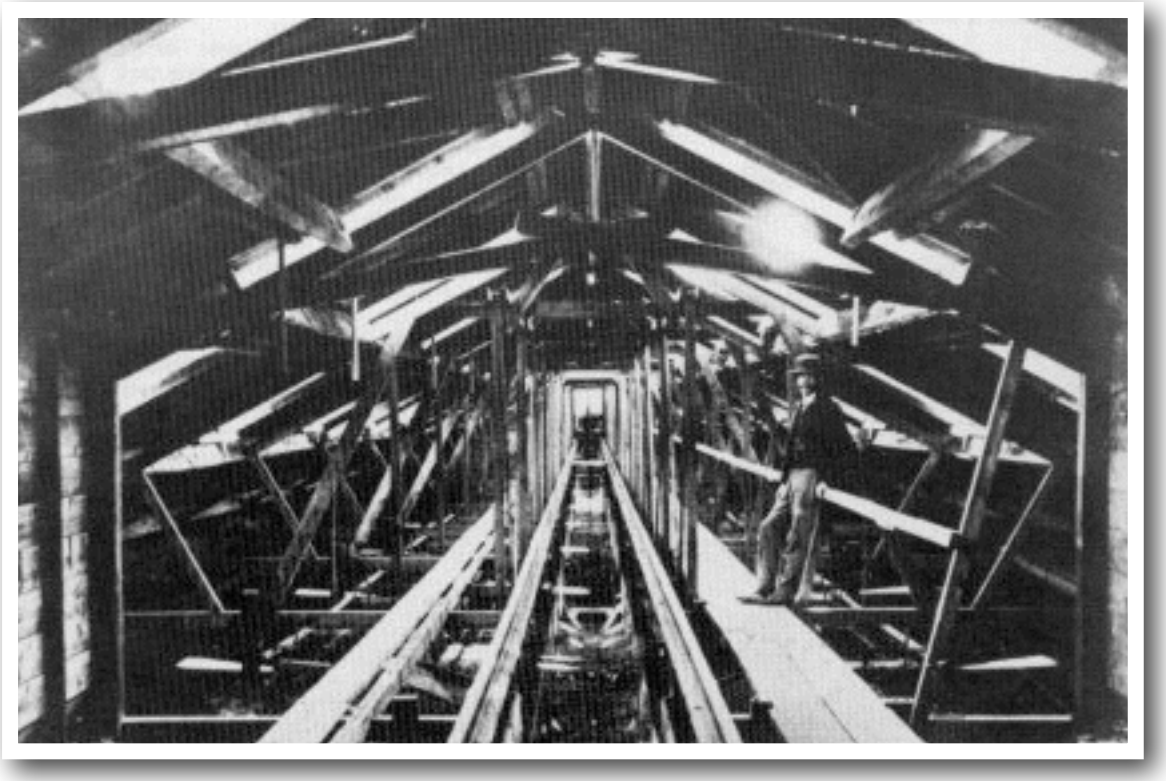
$V_{WAVE} = \sqrt{g\dfrac{\lambda}{2\pi}}$

λ = WAVELENGTH

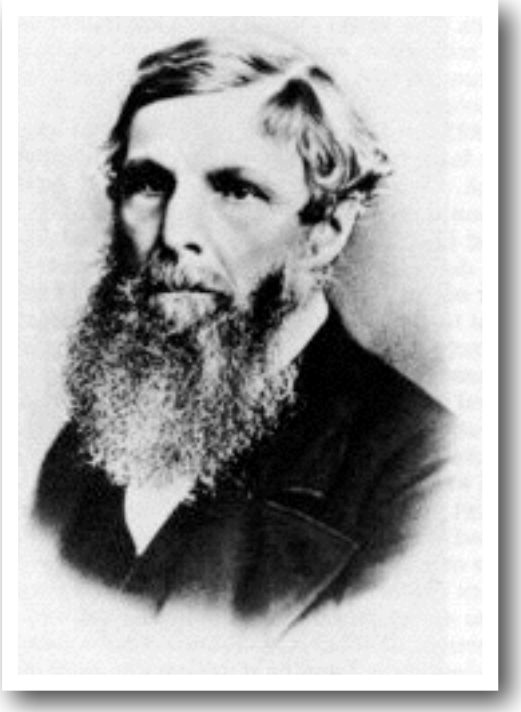
REYNOLDS NUMBER $= R_E \equiv \dfrac{\rho L^* U^*}{\mu}$

$= \dfrac{\text{INERTIAL TERM } (\bar{u}\cdot\bar{\nabla}\bar{u})}{\text{VISCOUS TERM } (\mu\nabla^2\bar{u})}$

William Froude

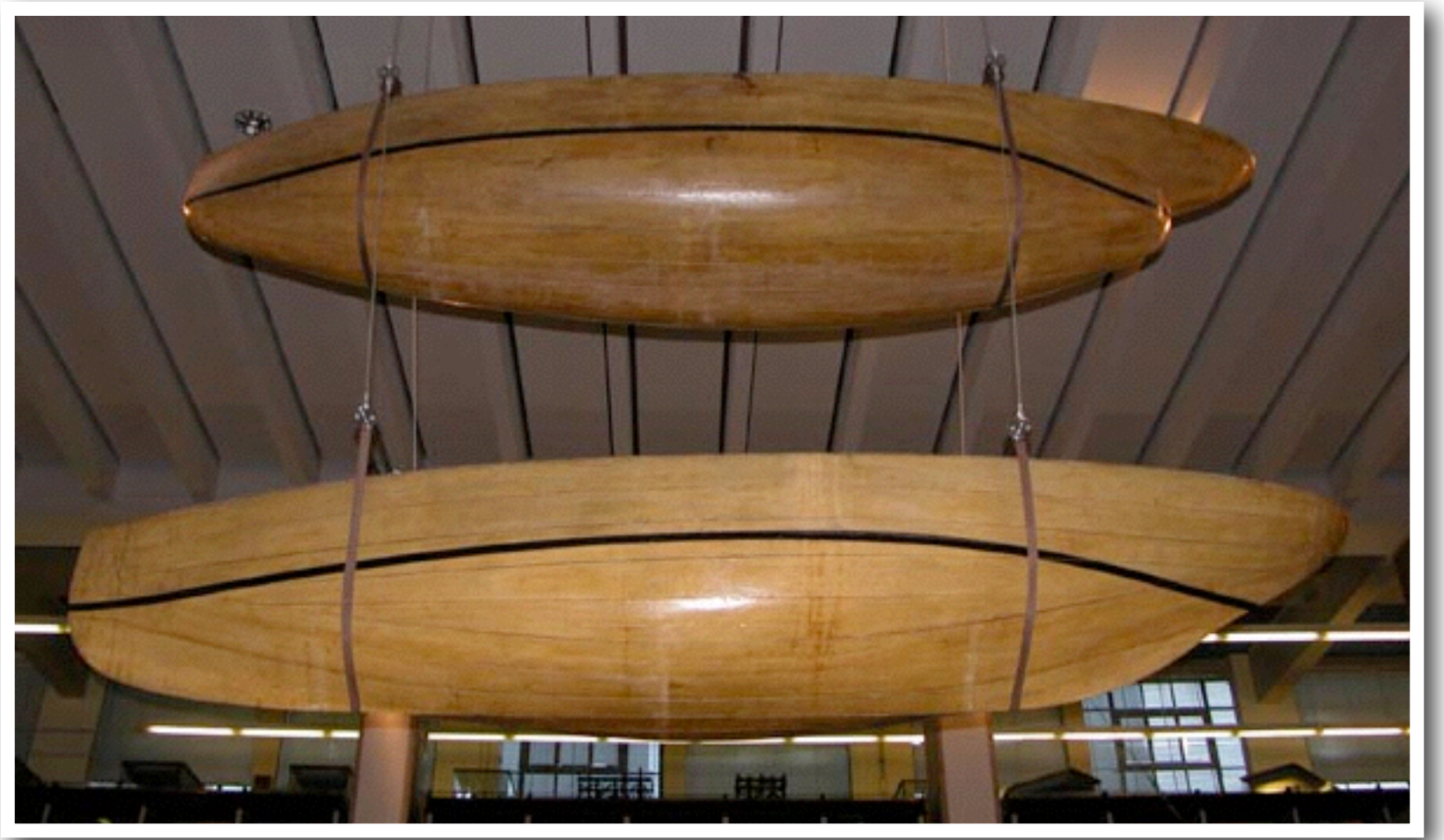


Chelston Cross Tank at Torquay Circa 1871



William Froude
(1810-1879)

Boat Models used by William Froude

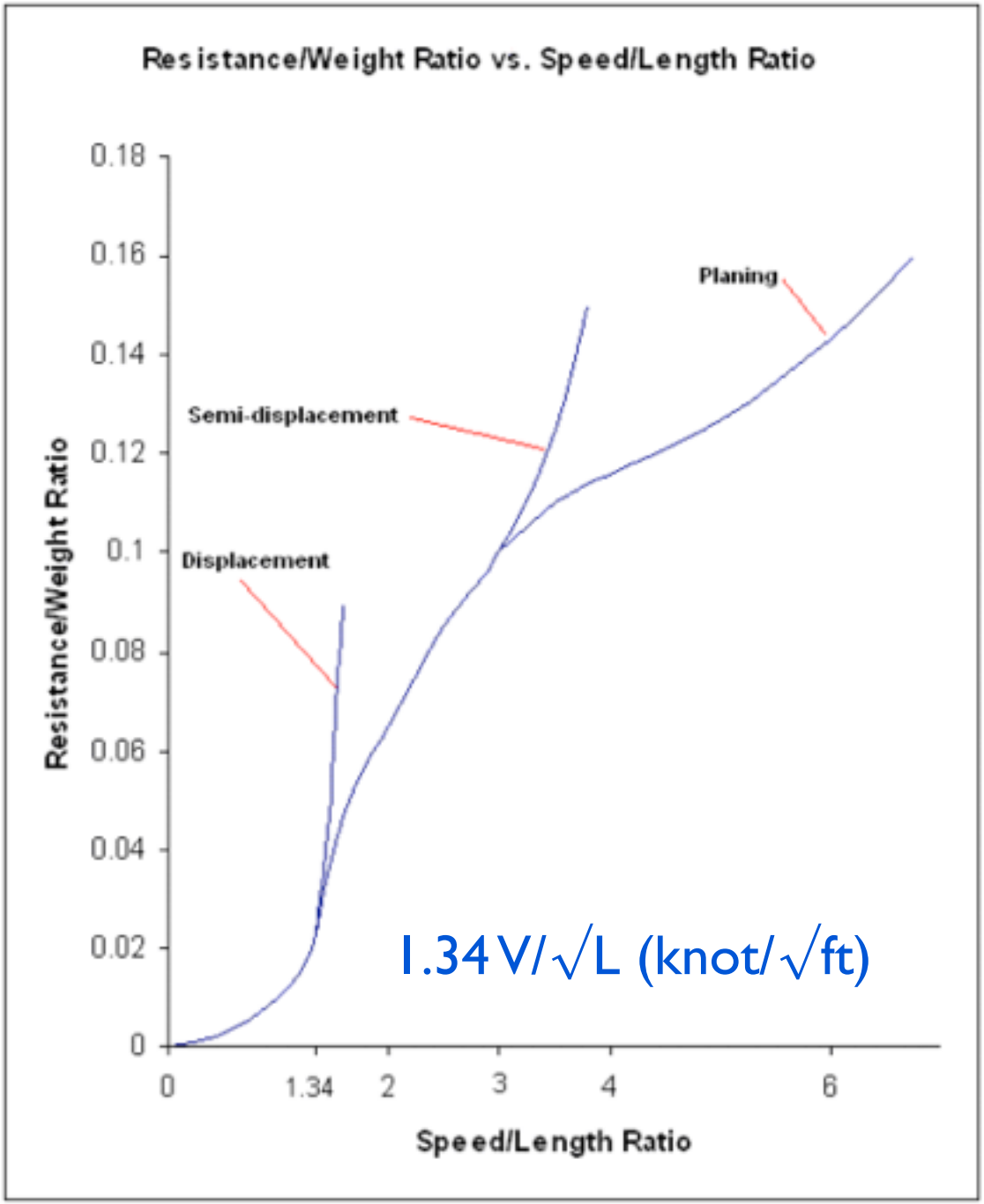


Boat Length = Speed^2

Boat Speed/ Length Ratios

- As speed (in knots) exceeds $1.34 \times \sqrt{(\text{length, ft})}$, then resistance increases exponentially.

- 1 knot ≈ 0.5 m/s, so critical Froude number is Fr ≈ 0.4

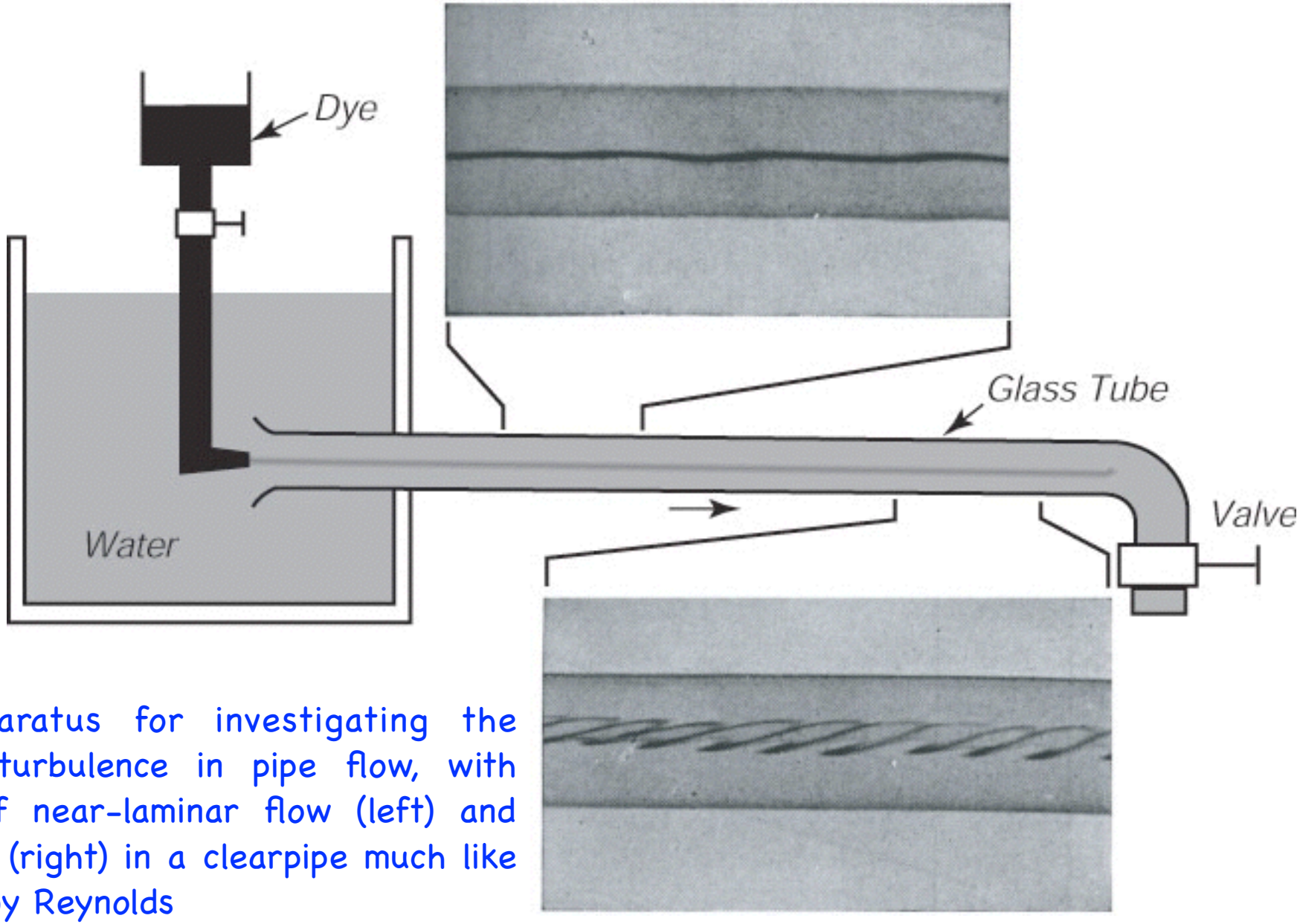


Osborn Reynolds

- Re $\equiv$ ρ LU/ μ
- Turbulence when Re > 3000
- Blood flow: Re ≈ 100
- Swimmer: Re ≈ 4,000,000
- HMS QE II: 5,000,000,000

(1842-1910)

Reynolds's Experiment



Reynolds apparatus for investigating the transition to turbulence in pipe flow, with photographs of near-laminar flow (left) and turbulent flow (right) in a clearpipe much like the one used by Reynolds

Drag Force on Sphere

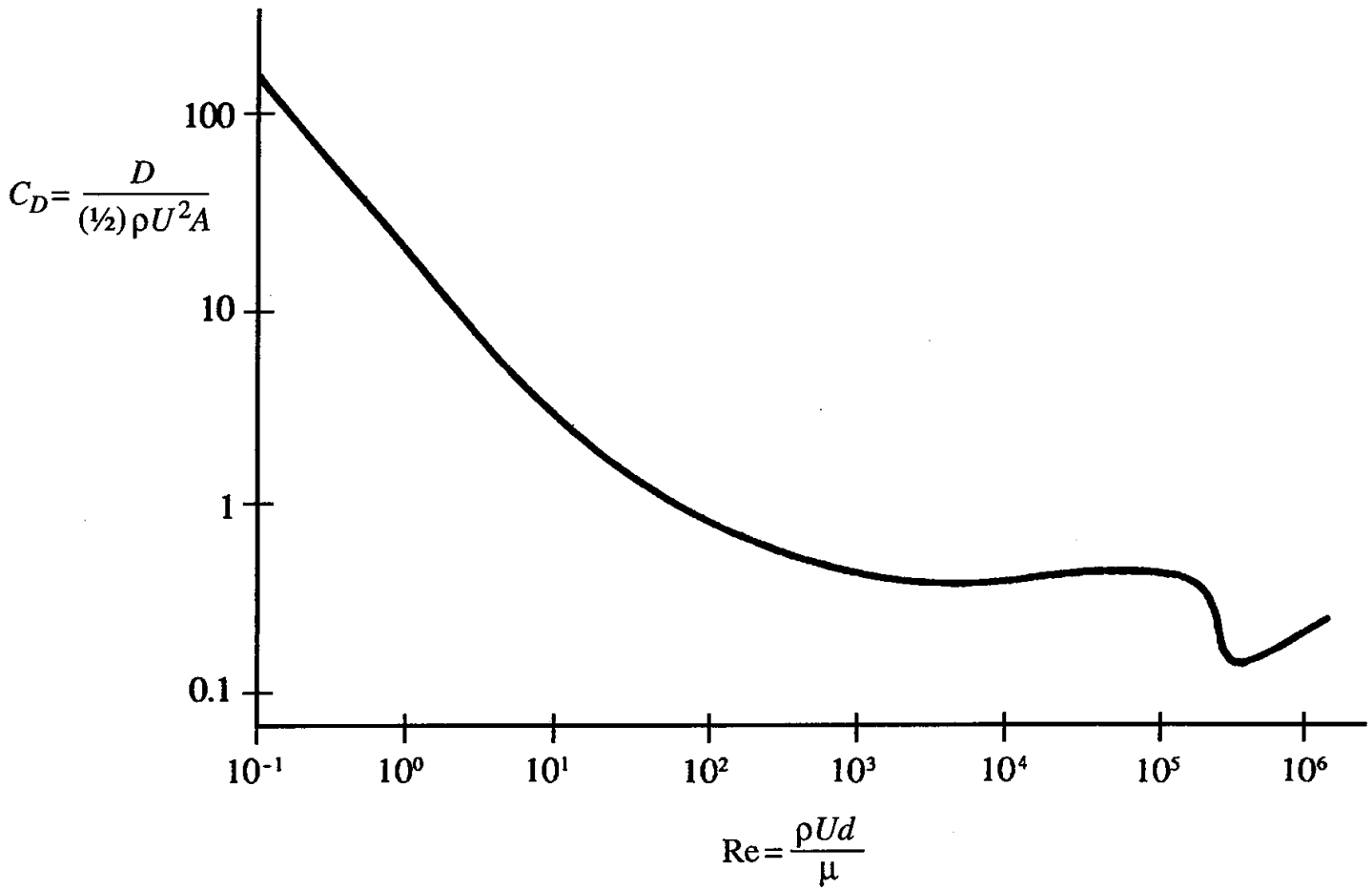


Figure 8.2 Drag coefficient for a sphere. The characteristic area is taken as $A = \pi d^2/4$. The reason for the sudden drop of C_D at Re $\sim 5 \times 10^5$ is the transition of the laminar boundary layer to a turbulent one, as explained in Chapter 10.

Drag Force on a Sphere

$$C_D = \frac{(\text{DRAG})}{\frac{1}{2}\rho U^2 A}$$

$\left(A \equiv \frac{1}{4}\pi d^2 \text{ FOR SPHERE}\right)$
$d = \text{DIAMETER}$

TWO LIMITS:

- STRONG VISCOSITY (NO INERTIAL EFFECTS)
- SMALL VISCOSITY (ONLY INERTIAL)

LOW RE (STRONG VISCOSITY)

$$\underset{\text{FORCE}}{(\text{DRAG})} \sim f(\mu, U, d)$$

$$[\mu] \sim \frac{\text{FORCE}}{LU}$$

$$\text{DRAG} \sim \mu d U$$

ONLY DIMENSIONALLY CORRECT COMBINATION.

THUS

$$\text{DRAG} \sim \left(\frac{1}{Re}\right)\rho U^2 A$$

HIGH Re (SMALL VISCOSITY)

$$(\text{DRAG}) \sim f(\rho, U, d)$$

$$\sim \rho U^2 A$$

INDEPENDENT OF Re !

$$Re = \frac{\rho L U}{\mu}$$

C_d for a flat plate

(Fig. 10.12)

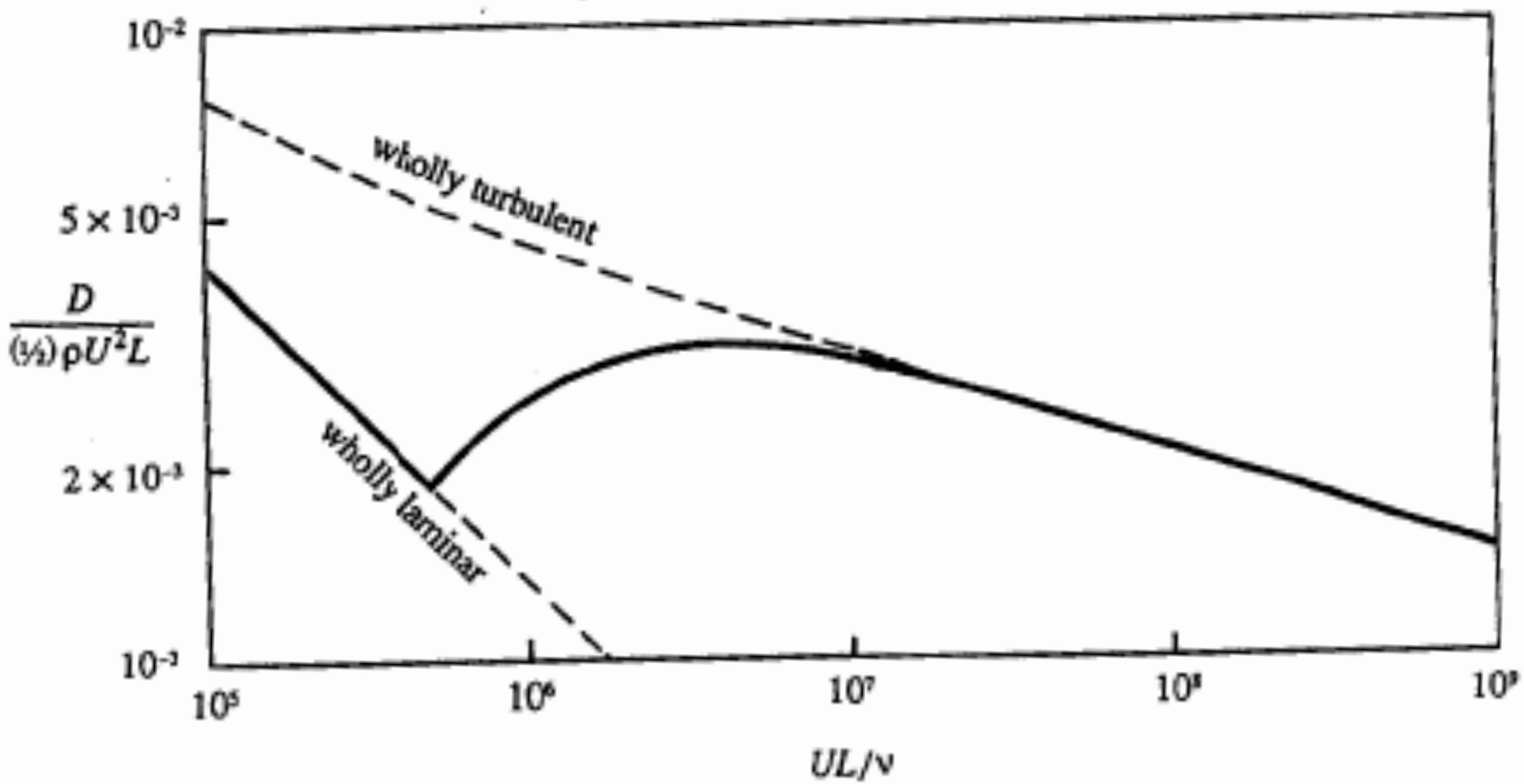


Figure 10.12 Measured drag coefficient for a boundary layer over a flat plate. The continuous line shows the drag coefficient for a plate on which the flow is partly laminar and partly turbulent, with the transition taking place at a position where the local Reynolds number is 5×10^5. The dashed lines show the behavior if the boundary layer was either completely laminar or completely turbulent over the entire length of the plate.

Dimensionless Numbers in Fluid Dynamics

Name(s)	Symbol	Definition	Significance
Alfvén, Kármán	Al, Ka	V_A/V	*(Magnetic force/ inertial force)$^{1/2}$
Bond	Bd	$(\rho' - \rho)L^2 g/\Sigma$	Gravitational force/ surface tension
Boussinesq	B	$V/(2gR)^{1/2}$	(Inertial force/ gravitational force)$^{1/2}$
Brinkman	Br	$\mu V^2/k\Delta T$	Viscous heat/conducted heat
Capillary	Cp	$\mu V/\Sigma$	Viscous force/surface tension
Carnot	Ca	$(T_2 - T_1)/T_2$	Theoretical Carnot cycle efficiency
Cauchy, Hooke	Cy, Hk	$\rho V^2/\Gamma = \mathrm{M}^2$	Inertial force/ compressibility force
Chandra-sekhar	Ch	$B^2 L^2/\rho\nu\eta$	Magnetic force/dissipative forces
Clausius	Cl	$LV^3\rho/k\Delta T$	Kinetic energy flow rate/heat conduction rate
Cowling	C	$(V_A/V)^2 = \mathrm{Al}^2$	Magnetic force/inertial force
Crispation	Cr	$\mu\kappa/\Sigma L$	Effect of diffusion/effect of surface tension
Dean	D	$D^{3/2}V/\nu(2r)^{1/2}$	Transverse flow due to curvature/longitudinal flow
[Drag coefficient]	C_D	$(\rho' - \rho)Lg/\rho' V^2$	Drag force/inertial force
Eckert	E	$V^2/c_p\Delta T$	Kinetic energy/change in thermal energy
Ekman	Ek	$(\nu/2\Omega L^2)^{1/2} = (\mathrm{Ro}/\mathrm{Re})^{1/2}$	(Viscous force/Coriolis force)$^{1/2}$
Euler	Eu	$\Delta p/\rho V^2$	Pressure drop due to friction/ dynamic pressure
Froude	Fr	$V/(gL)^{1/2}$ V/NL	†(Inertial force/gravitational or buoyancy force)$^{1/2}$
Gay–Lussac	Ga	$1/\beta\Delta T$	Inverse of relative change in volume during heating
Grashof	Gr	$gL^3\beta\Delta T/\nu^2$	Buoyancy force/viscous force
[Hall coefficient]	C_H	λ/r_L	Gyrofrequency/ collision frequency

*(†) Also defined as the inverse (square) of the quantity shown.

From NRL's Plasma Formulary

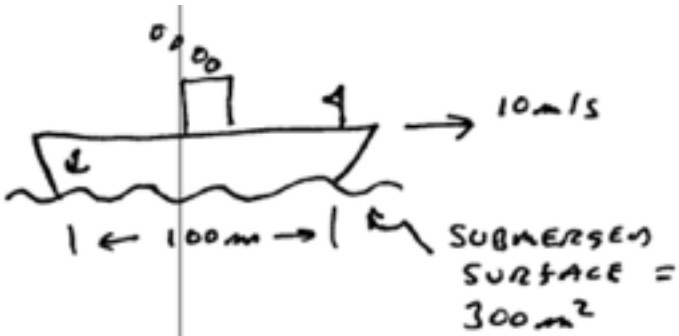
Dimensionless Numbers in Fluid Dynamics

(Page 2)

Name(s)	Symbol	Definition	Significance
Hartmann	H	$BL/(\mu\eta)^{1/2} = (\mathrm{Rm}\,\mathrm{Re}\,\mathrm{C})^{1/2}$	(Magnetic force/ dissipative force)$^{1/2}$
Knudsen	Kn	λ/L	Hydrodynamic time/ collision time
Lewis	Le	$\kappa/\mathcal{D}$	*Thermal conduction/molecular diffusion
Lorentz	Lo	V/c	Magnitude of relativistic effects
Lundquist	Lu	$\mu_0 L V_A/\eta = \mathrm{Al}\,\mathrm{Rm}$	$\mathbf{J}\times\mathbf{B}$ force/resistive magnetic diffusion force
Mach	M	V/C_S	Magnitude of compressibility effects
Magnetic Mach	Mm	$V/V_A = \mathrm{Al}^{-1}$	(Inertial force/magnetic force)$^{1/2}$
Magnetic Reynolds	Rm	$\mu_0 L V/\eta$	Flow velocity/magnetic diffusion velocity
Newton	Nt	$F/\rho L^2 V^2$	Imposed force/inertial force
Nusselt	N	$\alpha L/k$	Total heat transfer/thermal conduction
Péclet	Pe	LV/κ	Heat convection/heat conduction
Poisseuille	Po	$D^2\Delta p/\mu L V$	Pressure force/viscous force
Prandtl	Pr	ν/κ	Momentum diffusion/ heat diffusion
Rayleigh	Ra	$gH^3\beta\Delta T/\nu\kappa$	Buoyancy force/diffusion force
Reynolds	Re	LV/ν	Inertial force/viscous force
Richardson	Ri	$(NH/\Delta V)^2$	Buoyancy effects/ vertical shear effects
Rossby	Ro	$V/2\Omega L\sin\Lambda$	Inertial force/Coriolis force
Schmidt	Sc	$\nu/\mathcal{D}$	Momentum diffusion/ molecular diffusion
Stanton	St	$\alpha/\rho c_p V$	Thermal conduction loss/ heat capacity
Stefan	Sf	$\sigma L T^3/k$	Radiated heat/conducted heat
Stokes	S	$\nu/L^2 f$	Viscous damping rate/ vibration frequency
Strouhal	Sr	fL/V	Vibration speed/flow velocity
Taylor	Ta	$(2\Omega L^2/\nu)^2$	Centrifugal force/viscous force
		$R^{1/2}(\Delta R)^{3/2}\cdot(\Omega/\nu)$	(Centrifugal force/ viscous force)$^{1/2}$
Thring, Boltzmann	Th, Bo	$\rho c_p V/\epsilon\sigma T^3$	Convective heat transport/ radiative heat transport
Weber	W	$\rho L V^2/\Sigma$	Inertial force/surface tension

From NRL's Plasma Formulary

Example 8.1



WHAT IS THE MODEL SPEED AT 1/25 SCALE (i.e. 4 m LONG)?

MODEL DRAG IS 60 N. WHAT IS DRAG ON FULL-SCALE?

① MODEL SPEED

FROUDE NUMBERS SHOULD BE SAME

$$\frac{U_{model}}{\sqrt{g}\,\sqrt{\ell_m}} = \frac{10\ m/s}{\sqrt{g}\,\sqrt{\ell_f}} \Rightarrow U_{model} = 10\ m/s\sqrt{\frac{1}{25}} = 2\ m/s$$

② DRAG

TOTAL DRAG = (DRAG DUE TO VISCOSITY) + (WAVE DRAG DUE TO PUSHING WATER)

THIS IS SMALL, BUT WE SHOULD ESTIMATE

Viscous Drag on Hull

$$\text{VISCOUS DRAG} = \frac{1}{2} C_D \rho u^2 A$$

COEFFICIENT OF VISCOUS DRAG
DEPENDS ON REYNOLDS NUMBER

$$R_e = \frac{u\ell}{\nu} \qquad \nu = 10^{-6}\ m^2/s \text{ FOR WATER}$$

$$= \frac{2 \cdot 4}{10^{-6}} = 8 \times 10^6 \text{ (MODEL)}$$

$$= \frac{10 \cdot 100}{10^{-6}} = 1000 \times 10^6 \text{ (FULL-SCALE)}$$

C_D (MODEL) = 0.003

C_D (FULL SCALE) = 0.0015

$$\therefore \quad \text{VISCOUS DRAG} = \begin{cases} \frac{1}{2}\, 0.003\ 10^3\ 2^2 \left(\frac{300}{25^2}\right) = 2.9\ N \\ \frac{1}{2}\, 0.0015\ 10^3\ 10^4\ 300 = 23 \times 10^3\ N \end{cases}$$

Wave Drag

$$\text{WAVE DRAG (FULL SCALE)} = \text{WAVE DRAG (MODEL)} \times \left(\frac{\rho_F}{\rho_m}\right)\left(\frac{\ell_F}{\ell_m}\right)^2\left(\frac{U_F}{U_m}\right)^2$$

$$= (60 - 2.9\ N) \times (25)^2 \left(\frac{10}{2}\right)^2$$

$$= 9 \times 10^5\ N$$

15,625

SO TOTAL DRAG ON FULL SCALE

$$= 9 \times 10^5\ N + 0.27 \times 10^5\ N$$

$\sim$ 100 TONNES FORCE EQUIVALENT

Summary

- Dimensional analysis is a useful tool in many physical problems. Key scaling parameters can be identified and used to understand behaviors as size and velocity change.

- When the Reynolds number is not too large, flow is laminar. Some relatively simple problems can be solved analytically to guide our understanding of viscosity.