

# APPH 4200

# Physics of Fluids

## More 2D Potential Flow

## Chapter 6

1. More examples from Chapter 6: 2D, Inviscid, Irrotational flow
2. Blasius Theorem (Lift and Drag)
3. (Easy) CFD: potential flow in 2D

# Complex Velocity Potential

WHEN  $\nabla \times \bar{U} = \nabla \cdot \bar{U} = 0$  IN 2D, THEN WE  
CAN DEFINE

$\varphi$  = VELOCITY POTENTIAL

$\psi$  = STREAM FUNCTION (VECTOR POTENTIAL)

WHERE  $\nabla^2 \varphi = 0$

$\nabla^2 \psi = 0$

$$\frac{d\omega}{dz} = U_x - iU_y = \text{COMPLEX VELOCITY OF FLOW}$$
$$\text{REAL } \left( \frac{d\omega}{dz} \right) = U_x$$
$$\text{IMAGINARY } \left( \frac{d\omega}{dz} \right) = -U_y$$

SO A COMPLEX VELOCITY POTENTIAL IS A  
CONVENIENT MATHEMATICAL TECHNIQUE ("TRICK") TO  
SOLVE EULER POTENTIAL FLOW

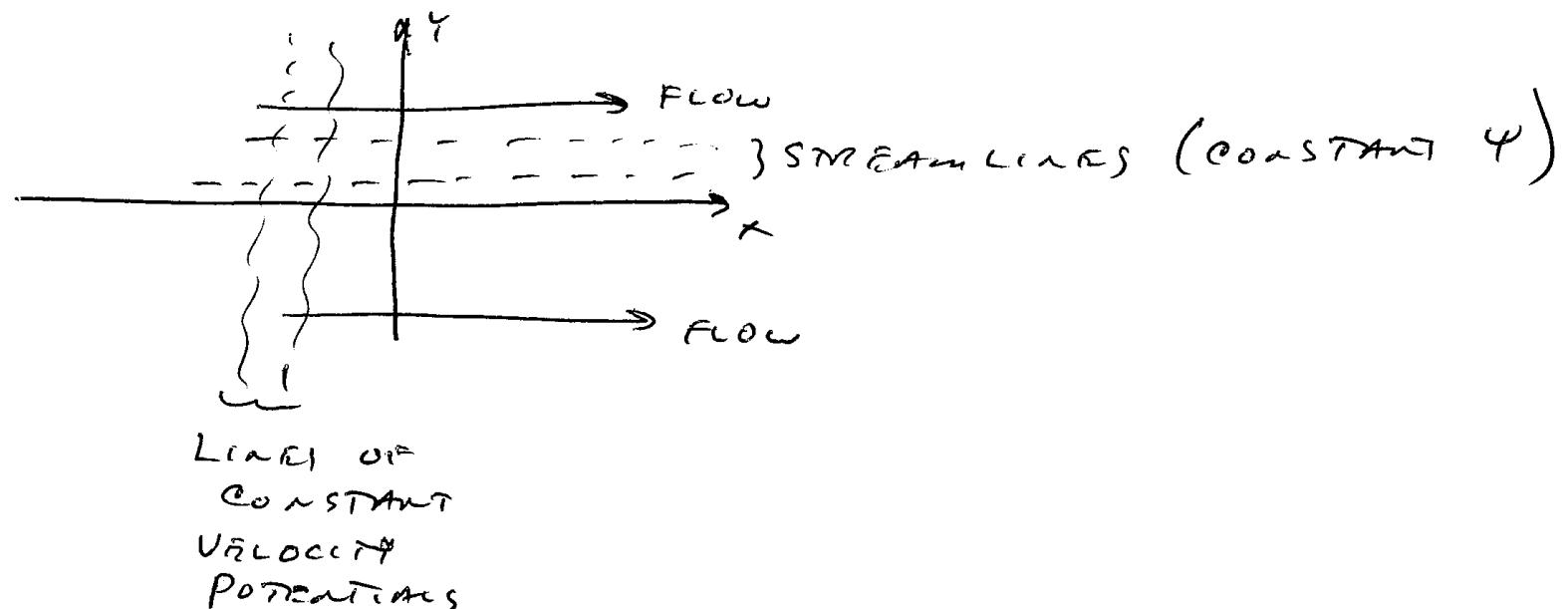
$$\omega(z) = \varphi + i\psi$$

ANY ANALYTIC FUNCTION  $\omega(z)$  IS A SOLUTION FOR  
2D POTENTIAL FLOW !!!

# Example Complex Potential

$$w(z) = Az = A(x + iy) \quad \varphi(x, y) = Ax \quad \psi(x, y) = Ay$$

$$\frac{dw}{dz} = A = U_x - iU_y \Rightarrow U_x = A = \text{constant}$$
$$U_y = 0$$



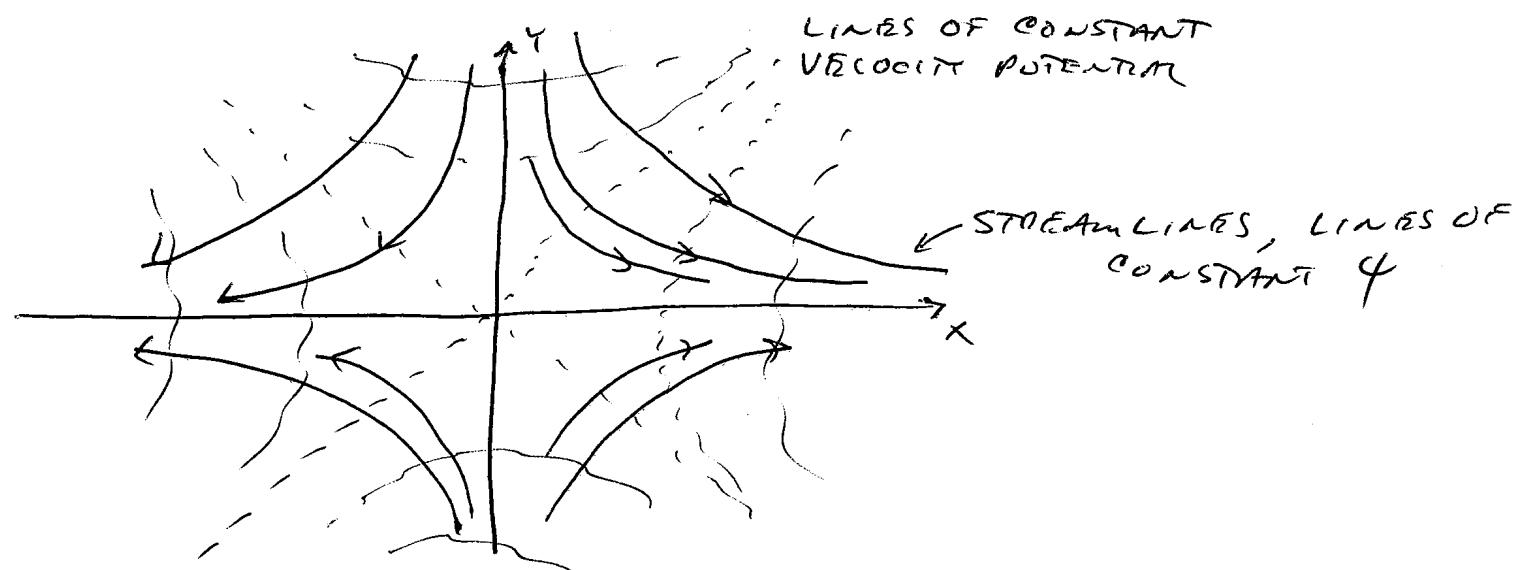
# Another Example

$$\omega(z) = A z^2 = A(x + iy)^2 = A(x^2 - y^2) + 2Ax i y$$

$$\varphi(x, y) = A(x^2 - y^2)$$

$$\psi(x, y) = 2Ax y$$

$$\frac{d\omega}{dz} = 2Az = 2A(x + iy) \Rightarrow u_x = 2Ax, u_y = -2Ay$$



NOTE:  $\frac{d\omega}{dz} \rightarrow 0$  AT  $z \rightarrow 0$ . THUS  $z = 0 = (x + iy)$  IS  
A STAGNATION POINT.

# Yet Another Example

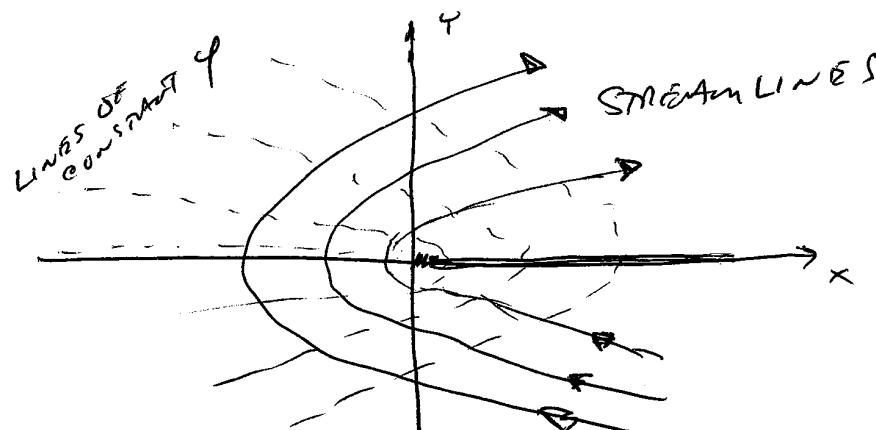
$$\omega(z) = A\sqrt{z} = A\sqrt{\rho} (\cos \theta/2 + i \sin \theta/2) \quad (\text{cylindrical coordinates})$$

$$\psi(r, \theta) = A\sqrt{r} \cos \theta/2$$

$$\psi(r, \theta) = A\sqrt{r} \sin \theta/2$$

$$\frac{d\omega}{dz} = \frac{A}{2} \frac{1}{\sqrt{z}} = \frac{A}{2\sqrt{r}} (\cos \theta/2 - i \sin \theta/2)$$

$$\Rightarrow u_x = \frac{A}{2\sqrt{r}} \cos \theta/2 \quad u_y = \frac{A}{2\sqrt{r}} \sin \theta/2$$



# Mass Source

$$\omega(z) = \frac{m}{2\pi} \ln z = \frac{m}{2\pi} \ln(r e^{i\theta}) = \frac{m}{2\pi} \left[ \ln r + i\theta \right]$$

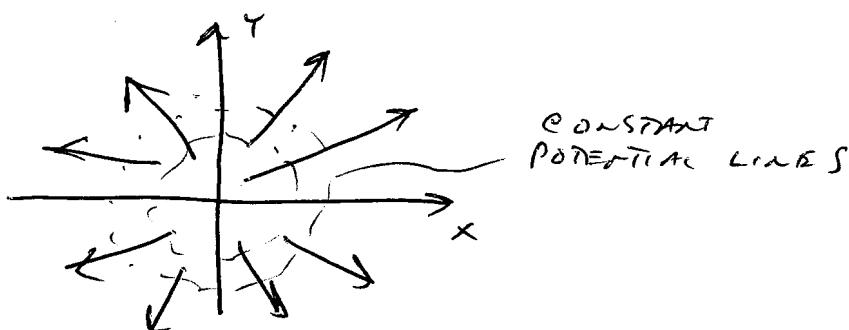
$$\varphi(r, \theta) = \frac{m}{2\pi} \ln r$$

$$\psi(r, \theta) = \frac{m}{2\pi} \theta$$

$$\begin{aligned} \frac{\partial \omega}{\partial z} &= \frac{m}{2\pi z} = \frac{m}{2\pi(x^2+y^2)}(x - iy) \Rightarrow U_x = \frac{m}{2\pi r} \left( \frac{x}{r} \right) & U_y = \frac{m}{2\pi r} \left( \frac{y}{r} \right) \\ &= \frac{m}{2\pi r} \cos \theta & = \frac{m}{2\pi r} \sin \theta \end{aligned}$$

$$\text{so } U_r(r, \theta) = \frac{m}{2\pi r}$$

$$U_\theta(r, \theta) = 0$$



# Line Vortex

$$\omega(z) = -i \frac{\Gamma}{2\pi} \ln z = -i \frac{\Gamma}{2\pi} [\ln r + i\theta]$$

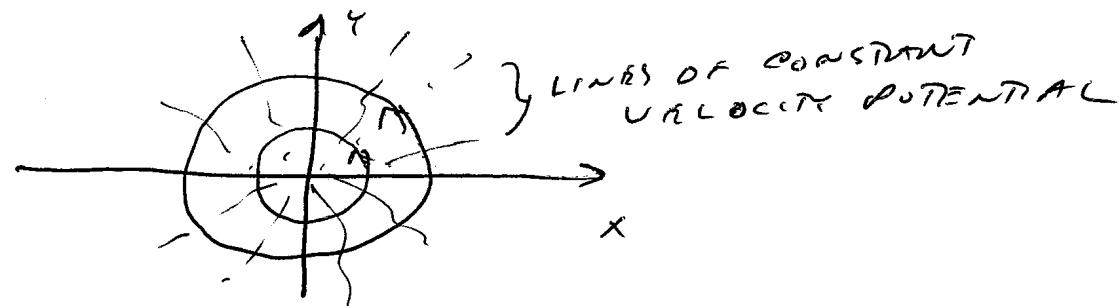
$$\psi(r, \theta) = \frac{\Gamma}{2\pi} \theta$$

$$\psi(r, \theta) = -\frac{\Gamma}{2\pi} \ln r$$

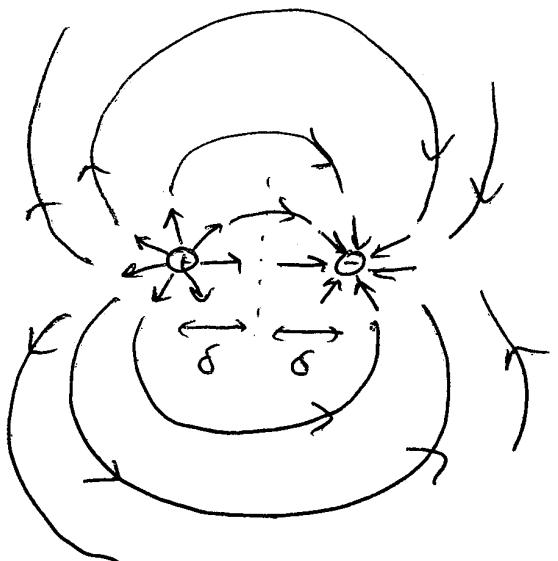
$$\frac{d\omega}{dz} = -i \frac{\Gamma}{2\pi} \frac{1}{z} = -i \frac{\Gamma}{2\pi r} e^{i\theta} (x - iy) = -i \frac{\Gamma}{2\pi r} \left( \frac{x}{r} \right) - \frac{\Gamma}{2\pi r} \left( \frac{y}{r} \right)$$

$$U_x = -\frac{\Gamma}{2\pi r} \sin \theta \quad U_y = \frac{\Gamma}{2\pi r} \cos \theta$$

$$\text{on} \quad U_r = 0 \quad U_\theta = \frac{\Gamma}{2\pi r}$$



# “Dipole” or Doublet



COMBINE EQUAL AND OPPOSITE MASS SOURCES ...

$$w(z) = \frac{M}{2\pi} \ln(z+\delta) - \frac{M}{2\pi} \ln(z-\delta)$$

As  $\delta \rightarrow 0$ , keep  $M\delta \sim \text{constant}$   
i.e.  $M \rightarrow \infty$

$$\begin{aligned} \text{THEN } w(z) &= \frac{m}{2\pi} \ln\left(\frac{z+\delta}{z-\delta}\right) = \frac{m}{2\pi} \ln\left(\frac{1 + \frac{\delta/z}{2}}{1 - \frac{\delta/z}{2}}\right) \\ &= \frac{m}{2\pi} \ln\left[\left(1 + \frac{\delta}{z}\right)\left(1 + \frac{\delta}{z} + \left(\frac{\delta}{z}\right)^2 + \dots\right)\right] \\ &\approx \frac{m}{2\pi} \ln\left[1 + \frac{2\delta}{z} + \dots\right] \\ &= \frac{(M\delta)}{\pi z} \quad \text{call } M\delta \equiv D = \text{constant} \end{aligned}$$

# Flow Past a Cylinder

(The "easy" complex way)

$$\text{LET } w(z) = U_z + \frac{D}{z} = U_r e^{i\theta} + \frac{D}{r} e^{-i\theta}$$

$$= \underbrace{\left( U_r + \frac{D}{r} \right) \cos\theta}_{\varphi(r, \theta)} + i \underbrace{\sin\theta \left( U_r - \frac{D}{r} \right)}_{-\psi(r, \theta)}$$

NOTE:

$$U_r = \frac{\partial \varphi}{\partial r} = \left( U - \frac{D}{r^2} \right) \cos\theta$$

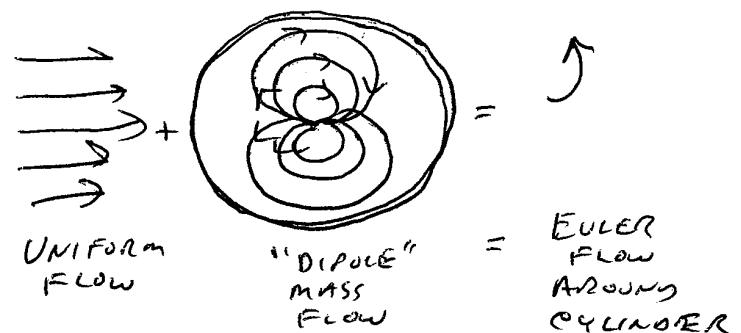
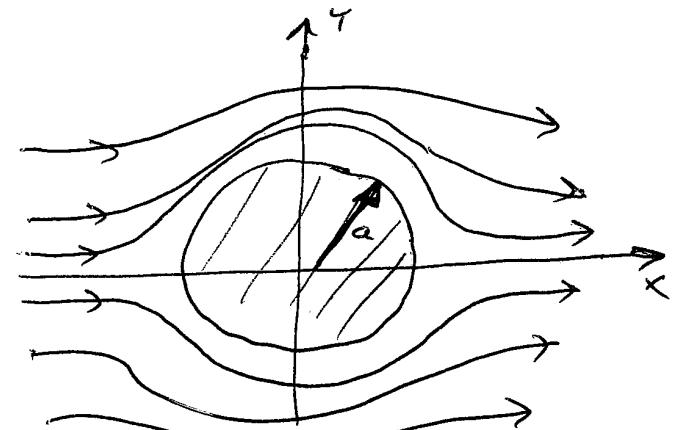
$$U_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} = - \left( U + \frac{D}{r^2} \right) \sin\theta$$

$$\text{BUT } U_r(r=a) = 0, \quad D = U a^2$$

so

$$U_r(r, \theta) = U \left( 1 - \left( \frac{a}{r} \right)^2 \right) \cos\theta$$

$$U_\theta(r, \theta) = -U \left( 1 + \left( \frac{a}{r} \right)^2 \right) \sin\theta$$



# Bernoulli's Principle

$$P + \frac{1}{2} \rho U^2 = \text{CONSTANT}$$

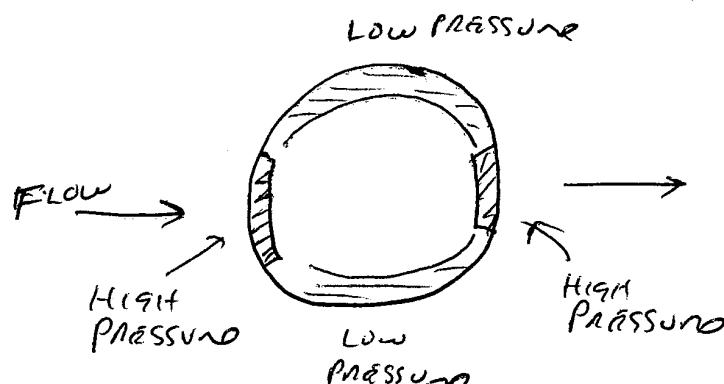
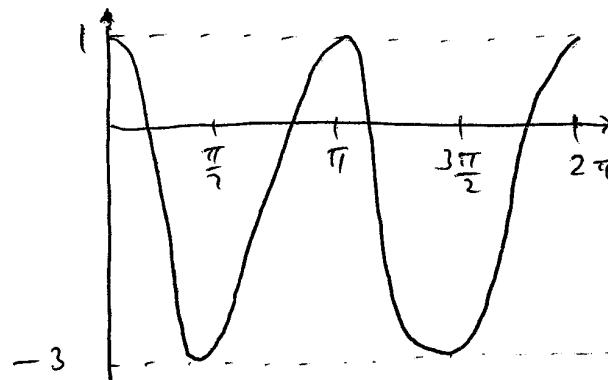
ALONG SURFACE OF CYLINDER  $U = U_0 = 4U^2 \sin^2 \theta$

THUS

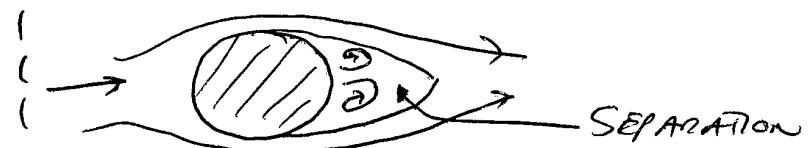
$$\begin{aligned} P_\infty + \frac{1}{2} \rho U^2 &= P(\theta) + \frac{1}{2} \rho U^2 \sin^2 \theta \\ &= P^* + 4U_p^2 \sin^2 \theta \end{aligned}$$

OR

$$\frac{P^* - P_\infty}{\frac{1}{2} \rho U^2} = 1 - 4 \sin^2 \theta$$



NOTE: WHEN A REAL FLUID FLOWS PAST A CYLINDER, THERE IS A NET DRAG FORCE. REASON: THE BACK-SIDE FLOW SEPARATES



# Blasius Theorem

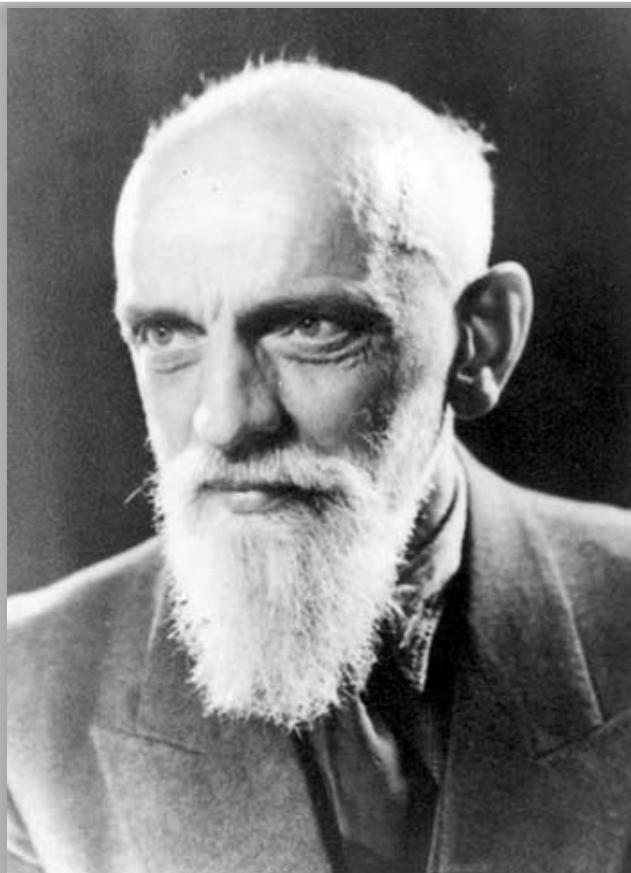


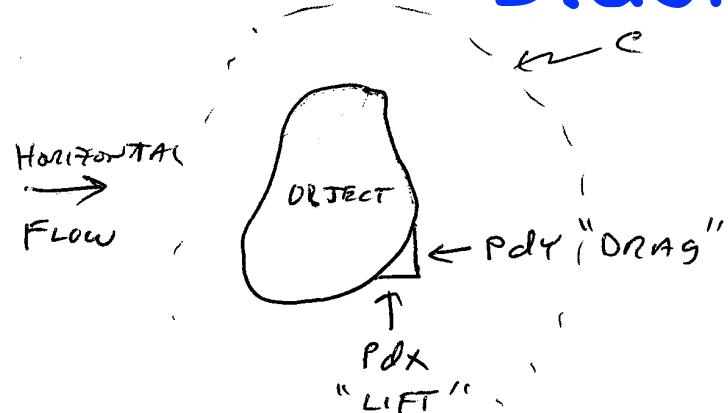
Fig. 1. Blasius in 1962, after retiring



Fig. 6. Blasius during lecturing, 1925

Paul Richard Heinrich Blasius (1883 - 1970) was one of the first students of Prandtl who provided a mathematical basis for boundary-layer drag (Chapter 10) but also showed as early as 1911 that the resistance to flow through smooth pipes could be expressed in terms of the Reynolds number for both laminar and turbulent flow.

# Blasius Theorem



$$dD - i dL = (\text{DRAG}) - i (\text{LIFT})$$

$$= -\rho dY - i \rho dX$$

$$= -i \rho \underbrace{dZ^*}_{\text{COMPLEX CONJUGATE}}$$

So

$$D - i L = -i \oint_C \rho dZ^*$$

BUT BERNOULLI gives  $P(z)$  ...

$$P(z) = P_\infty + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho (U_x^2 + U_y^2)$$

$$= P_\infty + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho (U_x + i U_y)(U_x - i U_y)$$

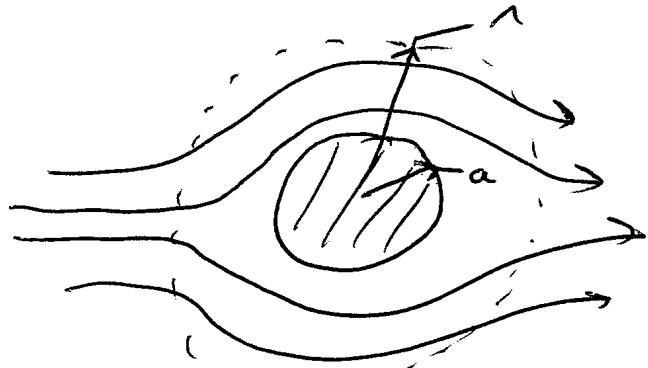
$$D - i L = -i \oint_C [P_\infty + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho (U_x + i U_y)(U_x - i U_y)] dZ^*$$

$$= +i \oint_C \frac{1}{2} \rho (U_x - i U_y) \underbrace{(U_x + i U_y)}_{|U| e^{i\Theta}} |dZ| e^{-i\Theta}$$

$\Rightarrow$  A REAL NUMBER!

$$= i \oint_C \frac{1}{2} \rho (U_x - i U_y)(U_x - i U_y) dZ = \underline{\underline{i \oint_C \frac{1}{2} \rho \left(\frac{dw}{dz}\right)^2 dz}}$$

## Example: Flow Past a Cylinder



$$w = U z + \frac{U a^2}{z}$$

$$D - iL = i \oint \frac{1}{2} \rho \left( \frac{dw}{dz} \right)^2 dz$$

$$\frac{dw}{dz} = U_x - i U_y$$

But  $U_x = U_1 \cos \theta - U_0 \sin \theta$   
 $U_y = U_1 \sin \theta + U_0 \cos \theta$

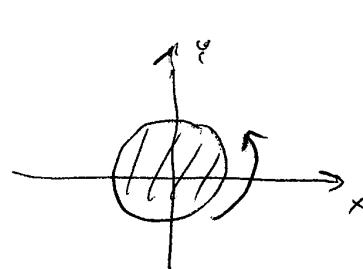
$$\frac{dw}{dz} = U - \frac{U a^2}{z^2}$$

$$\left( \frac{dw}{dz} \right)^2 = U^2 - \frac{2Ua^2}{z^2} + \frac{U^2 a^4}{z^4}$$

$$\oint \left( \frac{dw}{dz} \right)^2 dz = 0 \quad (\text{no residues}) \quad \therefore D = L = 0$$

# Flow Past a Rotating Cylinder

(Problem 6.9)



$$\omega(z) = U_z + \frac{Ua^2}{z} + \frac{i\Gamma}{2\pi} \ln\left(\frac{z}{a}\right)$$

$$\text{Since } \ln\left(\frac{z}{a}\right) = \ln\left(\frac{1}{a} e^{i\theta}\right) = \ln\left(\frac{1}{a}\right) + i\theta$$

$$\omega(z) = \left(U_r + \frac{Ua^2}{r}\right) \cos\theta - \frac{\Gamma\theta}{2\pi}$$

$$+ i \underbrace{\left[ \sin\theta \left(U_r - \frac{Ua^2}{r}\right) + \frac{\Gamma}{2\pi} \ln\left(\frac{1}{a}\right) \right]}_{\psi(r, \theta)}$$

$$\frac{d\omega}{dz} = U - \frac{Ua^2}{z^2} - \frac{i\Gamma}{2\pi z}$$

NOTE: Flow vanishes (STAGNATION POINT) when

$$\frac{d\omega}{dz} = 0 \Rightarrow U - \frac{Ua^2}{z^2} - i\frac{\Gamma}{2\pi z} = 0$$

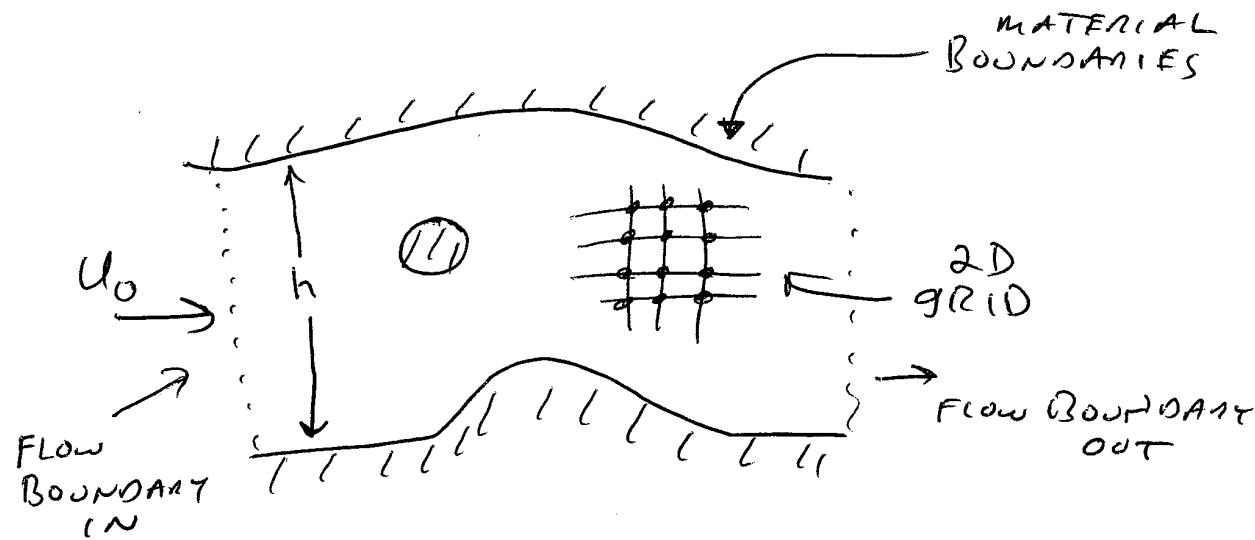
$$\text{or } \left(\frac{z}{a}\right)^2 - i\frac{\Gamma}{2\pi U a} \left(\frac{z}{a}\right) - 1 = 0 \Rightarrow \left(\frac{z}{a}\right) = i\frac{\Gamma}{4\pi U a} \pm \sqrt{1 - \left(\frac{\Gamma}{4\pi U a}\right)^2}$$

BLAISUS THEOREM:

$$(BRA) - i(CLF) = i \oint \frac{1}{2} g \left( \frac{d\omega}{dz} \right)^2 dz = i \oint \frac{1}{2} g \left[ \dots + i \frac{U\Gamma}{\pi z} + \dots \right] dz$$

$\underset{\text{Residue}}{\int}$

# Numerical Solution to Potential Flow in 2D



$$\nabla^2 \psi = 0$$

STREAM FUNCTION

$$\nabla^2 \varphi = 0$$

VELOCITY POTENTIAL

$$U_x = \frac{\partial \psi}{\partial y}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$U_y = -\frac{\partial \psi}{\partial x}$$

WITHIN Computational Domain: FINITE DIFFERENCE

$$\psi(\text{in/out}, y) \approx U_0 y$$

$$\psi(\text{bottom wall}) \approx 0$$

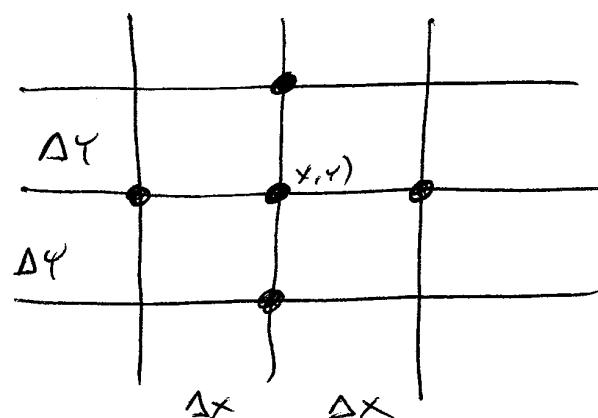
$$\psi(\text{top wall}) \approx U_0 \text{ height}$$

# Finite Difference Approximation

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

"CENTRAL DIFFERENCE  
MORE ACCURATE"

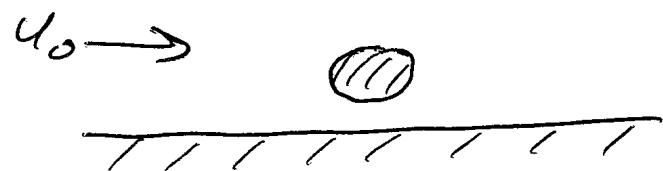
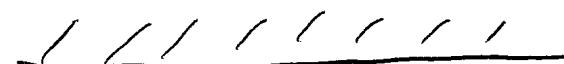
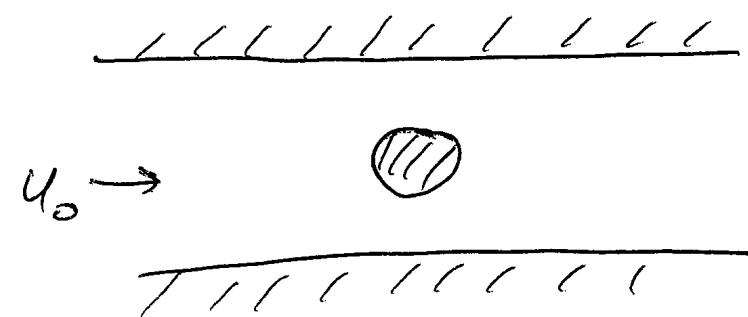
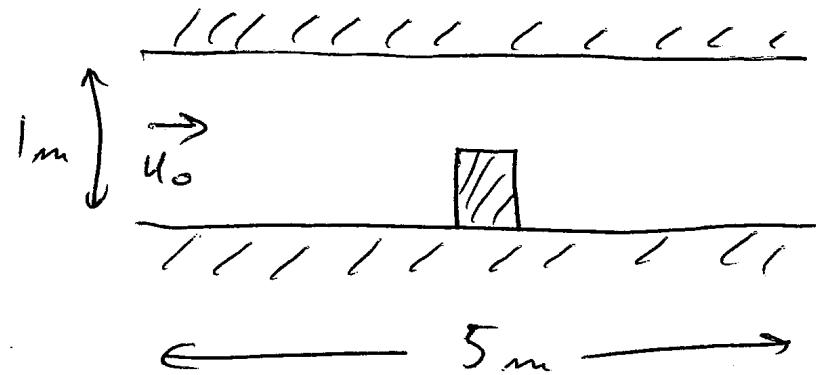
$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &\approx \frac{\frac{\partial u}{\partial x}\Big|_{x+\frac{\Delta x}{2}} - \frac{\partial u}{\partial x}\Big|_{x-\frac{\Delta x}{2}}}{\Delta x} \\ &\approx \frac{(u(x+\Delta x) - u(x)) - (u(x) - u(x-\Delta x))}{(\Delta x)^2} \\ &\approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2} \end{aligned}$$



LAPLACE'S EQUATION:

$$\begin{aligned} & \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2} \\ & + \frac{u(y+\Delta y) - 2u(y) + u(y-\Delta y)}{(\Delta y)^2} = 0 \end{aligned}$$

# Three Examples



Mathematica is available at all Columbia computer labs.  
See <http://www.columbia.edu/acis/facilities/software.html>

# Problems in Ch 6

- “Doublet” (or dipole) potential
- Drag on a “half-body”
- Potential flow around an ellipsoid

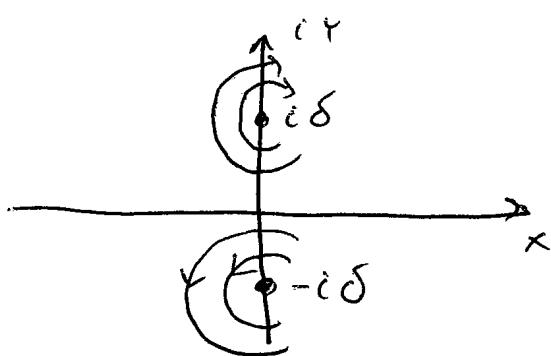
# Problem 6.1

1. In Section 7, the doublet potential

$$w = \mu/z,$$

was derived by combining a source and a sink on the  $x$ -axis. Show that the same potential can also be obtained by superposing a clockwise vortex of circulation  $-\Gamma$  on the  $y$ -axis at  $y = \varepsilon$ , and a counterclockwise vortex of circulation  $\Gamma$  at  $y = -\varepsilon$ , and letting  $\varepsilon \rightarrow 0$ .

# Problem 6.1

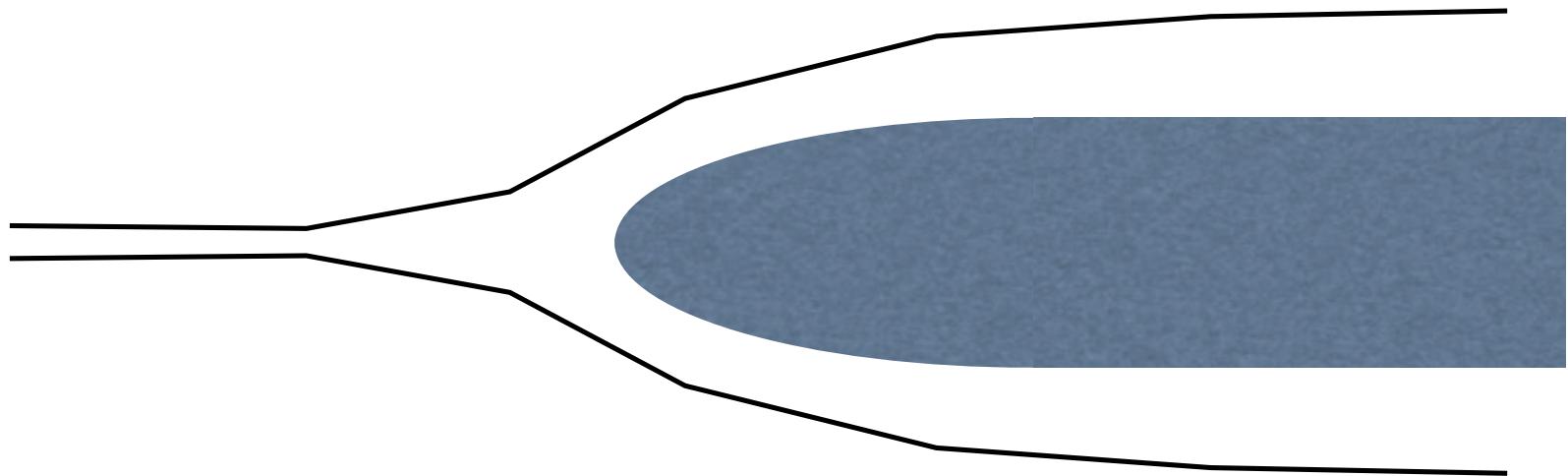


$$\begin{aligned}
 w(z) &= -i \frac{R}{2\pi} \ln(z - i\delta) + i \frac{R}{2\pi} \ln(z + i\delta) \\
 &= i \frac{R}{2\pi} \ln \left( \frac{z+i\delta}{z-i\delta} \right) \\
 &\approx i \frac{R}{2\pi} \ln \left[ \left( 1 + \frac{i\delta}{z} \right) \left( 1 + i \frac{\delta}{z} - \frac{\delta^2}{z^2} + \dots \right) \right] \\
 &\approx i \frac{R}{2\pi} \ln \left[ 1 + i \frac{2\delta}{z} \dots \right] \\
 &\approx - \frac{R\delta}{\pi z}
 \end{aligned}$$

$\therefore$  EXACTLY THE SAME FORM AS A  
 POINT SOURCE-SINK DOUBLET PAIR

# Problem 6.2

2. By integrating pressure, show that the drag on a plane half-body (Section 8) is zero.



# Problem 6.2

$$\text{DRAG} \sim \oint_0^{2\pi} C_p d\theta$$

$$\frac{dw}{dz} = U + \frac{m}{2\pi z}$$

STAGNATION:  $z = -\frac{m}{2\pi U}$

$$\psi_{\text{STAGNATION}} = \frac{m}{2} (\theta = \pi)$$

So BODY SHAPE given by

$$\boxed{U_1 \sin \theta + \frac{m}{2\pi} \theta = \frac{m}{2}}$$

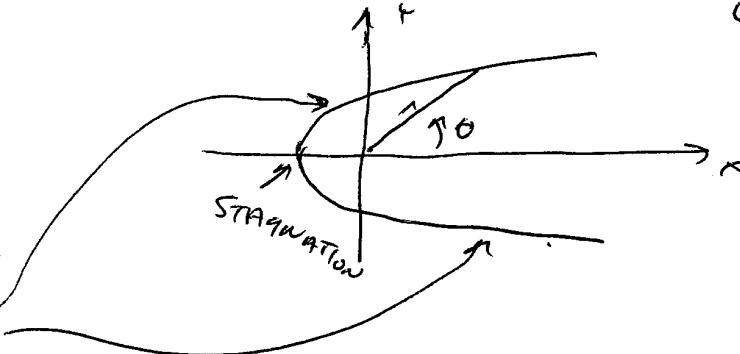
$$w(z) = Uz + \frac{m}{2\pi} \ln(z)$$

$$= U_1 e^{i\theta} + \frac{m}{2\pi} \ln(e^{i\theta})$$

$$= (U_1 \cos \theta + \frac{m}{2\pi} \ln 1)$$

$$+ i \left[ U_1 \sin \theta + \frac{m}{2\pi} \theta \right]$$

$$\psi(r, \theta)$$



$$C_p = 1 - \frac{U^2}{U_0^2}$$

$$U^2 = U_x^2 + U_\phi^2 = \left| \frac{dw}{dz} \right|^2 = \left( U + \frac{m}{2\pi z} \right) \left( U + \frac{m}{2\pi z} \right) = U^2 + \frac{mU}{2\pi} \underbrace{\left( \frac{1}{z} + \frac{1}{z^2} \right)}_{2 \cos \theta} + \frac{m^2}{(2\pi)^2 z^2}$$

$$\therefore C_p = -\frac{m}{2\pi U} \left[ \frac{m}{2\pi U_1 r^2} + \frac{2 \cos \theta}{r} \right]$$

## Problem 6.2 (cont)

BUT SURFACE OF BODY IS  $\frac{m \sin \theta + \frac{m}{2\pi} \theta}{2} = \frac{m}{2}$

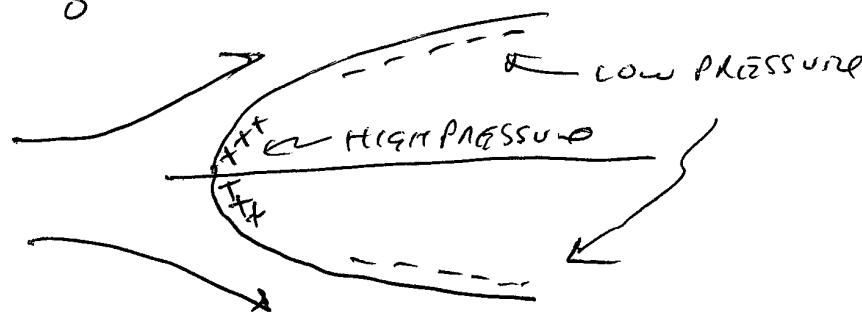
$$\text{OR } 1 = \frac{m(\pi - \theta)}{2\pi m \sin \theta} \quad \text{OR} \quad \frac{m}{2\pi m} = \frac{\pi \sin \theta}{(\pi - \theta)}$$

THUS

$$\begin{aligned} C_p &= -\frac{\sin \theta}{\pi - \theta} \left[ \frac{\sin \theta}{\pi - \theta} + 2 \cos \theta \right] \\ &= -\frac{d}{d\theta} \left[ (\pi - \theta) \left( \frac{\sin \theta}{\pi - \theta} \right)^2 \right] \\ &= -\left[ \left( \frac{\sin \theta}{\pi - \theta} \right)^2 + (\pi - \theta) 2 \left( \frac{\sin \theta}{\pi - \theta} \right) \underbrace{\frac{d}{d\theta} \left( \frac{\sin \theta}{\pi - \theta} \right)}_{\left( \frac{\cos \theta}{\pi - \theta} + \frac{\sin \theta}{(\pi - \theta)^2} \right)} \right] \end{aligned}$$

THUS

$$\int_0^{2\pi} C_p d\theta = 0 \quad \text{NO DRAG!}$$



# Problem 6.4

(Rankin Ovoid, ca. 1871)

4. Take a plane source of strength  $m$  at point  $(-a, 0)$ , a plane sink of equal strength at  $(a, 0)$ , and superpose a uniform stream  $U$  directed along the  $x$ -axis. Show that there are two stagnation points located on the  $x$ -axis at points

$$\pm a \left( \frac{m}{\pi a U} + 1 \right)^{1/2}.$$

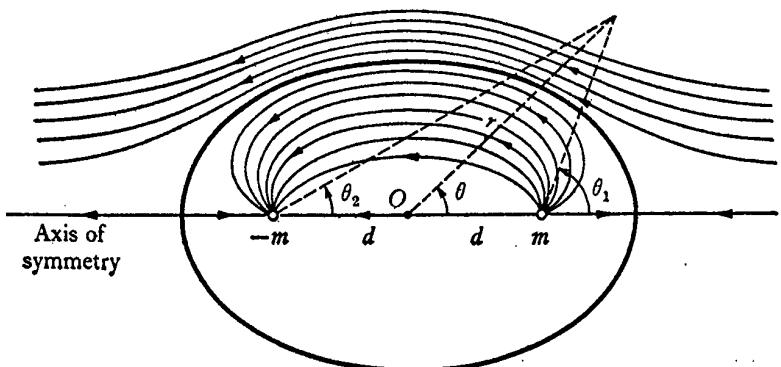
Show that the streamline passing through the stagnation points is given by  $\psi = 0$ . Verify that the line  $\psi = 0$  represents a closed oval-shaped body, whose maximum width  $h$  is given by the solution of the equation

$$h = a \cot \left( \frac{\pi U h}{m} \right).$$

The body generated by the superposition of a uniform stream and a source–sink pair is called a *Rankine body*. It becomes a circular cylinder as the source–sink pair approach each other.

# Problem 6.4

(Rankin Ovoid, ca. 1871)



$$W(z) = \frac{m}{2\pi} \ln(z-a) - \frac{m}{2\pi} \ln(z+a) - Uz$$

$$\begin{aligned} \frac{dW(z)}{dz} &= \frac{m}{2\pi(z-a)} - \frac{m}{2\pi(z+a)} - U \\ &= \frac{m}{\pi} \frac{a}{(z^2-a^2)} - U \end{aligned}$$

STAGNATION:  $\frac{dW}{dz} = 0 \quad \text{OR}$

$$z^2 - a^2 = \frac{m a}{\pi U}$$

$$z = \pm a \sqrt{1 + \frac{m}{\pi U}}$$

STREAM FUNCTION IS  $\psi = 0$  AT

STAGNATION BECAUSE STAGNATION POINTS LIE ON X-AXIS (REAL)

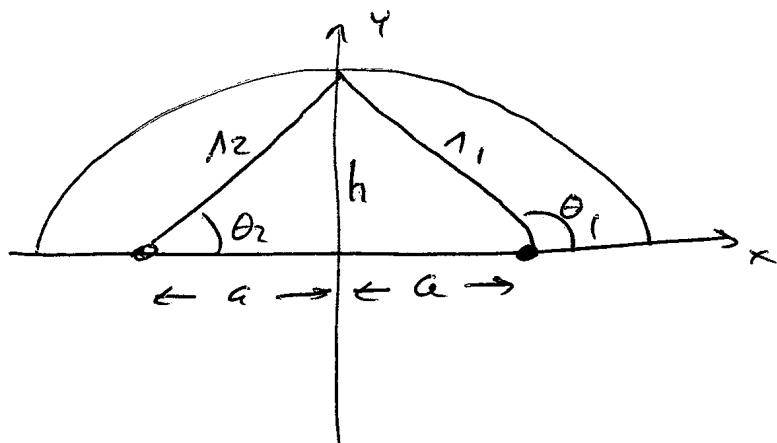
Ans  $I_m(\omega) = 0 = \psi(x, r)$

HEIGHT OF OVOID:  $I_m(\omega(z)) = 0$  AT  $x=0, \theta = \pm \frac{\pi}{2}, iY = \pm h$

$$\begin{aligned} W(z) &= \frac{m}{2\pi} \ln(r_1 e^{i\theta_1}) - \frac{m}{2\pi} \ln(r_2 e^{i\theta_2}) - U_r r e^{i\theta} \\ &= \left[ \frac{m}{2\pi} \ln r_1 - \frac{m}{2\pi} \ln r_2 - U_r \cos \theta \right] + i \underbrace{\left( \frac{m}{2\pi} (\theta_1 - \theta_2) - U_r \sin \theta \right)}_{-i\psi} \end{aligned}$$

# Problem 6.4 (cont)

To find the height



$$Uh = \frac{m}{2\pi} (\theta_1 - \theta_2)$$

$$\text{BUT } \theta_1 = \pi - \theta_2$$

$$\theta_2 = \tan^{-1} \left( \frac{h}{a} \right)$$

$$\text{So } Uh = \frac{m}{2\pi} \left( \pi - 2 \tan^{-1} \left( \frac{h}{a} \right) \right)$$

$$\frac{\pi Uh}{m} = \frac{\pi}{2} - \tan^{-1} \left( \frac{h}{a} \right) = \cot^{-1} \left( \frac{h}{a} \right)$$

$$\text{or } \frac{h}{a} = \cot \operatorname{An} \left( \frac{\pi Uh}{m} \right)$$

# Summary

- The complex potential is defined on the complex  $z$ -plane ( $z = x + i y$ ) and contains both the velocity potential,  $\phi(x,y)$ , and the streamfunction,  $\Psi(x,y)$ , or vector potential.
- In 2D, potential flow is very quickly calculated using today's computers.