APPH 4200
Physics of Fluids

Potential Flow (Ch. 6)
2D Irrotational, Incompressible Flow
Oct 11, 2011

1. Irrotational Flow
2. **Potential Flow in 2D** (incompressible)
3. Complex Variables (quick review)
4. Examples

Irrotational Flow (Ch.6)

- ➡ Potential Flow
- ➡ Complex potential (and working with complex representation of 2D flow)
- Blasius Theorem/Kutta-Zhukhovsky Theorem
- Numerical Solution (finite difference)
- Axisymmetric Flow

Irrotational Flow is Commonplace

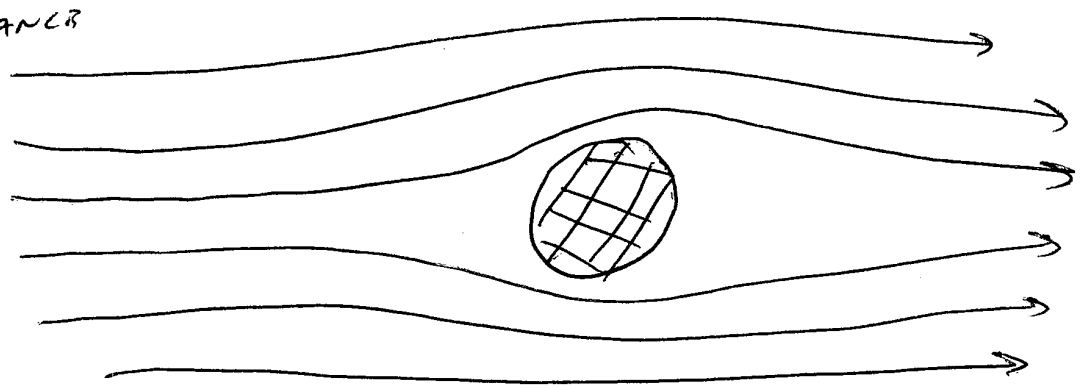
KELVIN'S THEOREM STATES THAT $\iint \bar{\Omega} \cdot d\bar{A} = \oint \bar{u} \cdot d\bar{l} = \Gamma$ IS A CONSTANT FOR A SURFACE MOVING WITH THE FLUID.

THIS IS A POWERFUL TOOL FOR ANALYZING FLOW PAST OBJECTS.

AT DISTANCE

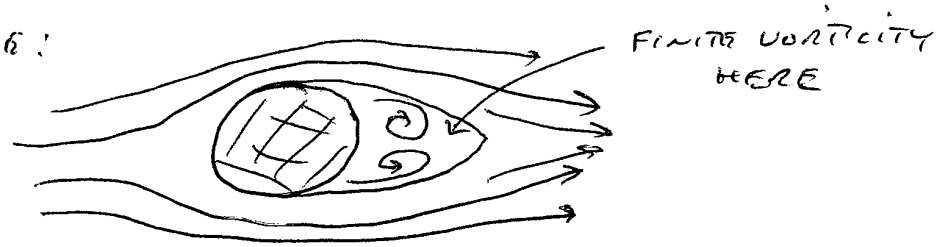
$\nabla \times \bar{u} = 0$

$\bar{\Omega} = 0$



SINCE $\bar{\Omega} = 0$ AT A DISTANCE, $\bar{\Omega} = 0$ EVERYWHERE ALONG STREAMLINES (PATHLINES)!

NOTE: THE EXCEPTION OCCURS WHEN THE VISCOUS BOUNDARY LAYER SEPARATES AT SURFACE:



Irrotational Flow Consequences

- IRROTATIONAL FLOW HAS NO <u>CLOSED</u> STREAMLINES. SINCE $\bar{u}$ IS ALWAYS PARALLEL TO $d\bar{l}$ AND

$$\oint \bar{u} \cdot d\bar{l} = 0$$

- SINCE $\nabla \times \bar{u} = 0$, THEN VELOCITY CAN BE EXPRESSED AS A GRADIENT OF A POTENTIAL

$$\bar{u} = \nabla \varphi \qquad \nabla \times \nabla \varphi = 0$$

- EULER'S EQUATION CAN BE SIMPLIFIED...

$$\frac{\partial \bar{u}}{\partial t} + \bar{\nabla}\left(\tfrac{1}{2}u^2\right) - \bar{u} \times (\nabla \times \bar{u}) = -\nabla P/\rho$$

OR

$$\nabla \underbrace{\left(\frac{\partial \varphi}{\partial t} + \frac{1}{2}u^2 + \frac{P}{\rho} \right)}_{} = 0$$

THUS, THIS FUNCTION IS CONSTANT/EQUAL <u>EVERYWHERE</u> IN SPACE/FLOW. MAY BE A FUNCTION OF TIME

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}u^2 + P/\rho = \text{CONSTANT IN SPACE} = f(t)$$

Potential Flow: Incompressible in 2D

CONDITIONS FOR INCOMPRESSIBLE FLOW

- $|u| \ll$ (SPEED OF SOUND)
- LENGTH $\ll$ (SPEED OF SOUND) / CHARACTERISTIC FREQUENCY

THE LAST CONDITION MEANS THAT SOUND PROPAGATES ACROSS FLOW MUCH FASTER THAN TIME VARIATION OF FLOW.

$\nabla \cdot \bar{u} = 0$ MEANS $\bar{u} = -\nabla z \times \nabla \psi$ FOR 2D FLOW

z = SYMMETRY DIRECTION

ψ = STREAM FUNCTION

$$\nabla \cdot (\nabla z \times \nabla \psi) = \nabla \psi \cdot (\nabla \times \nabla z) - \nabla z \cdot (\nabla \times \nabla \psi) = 0$$

Potential Flow in 2D

INCOMPRESSIBLE

$$\nabla \cdot \bar{u} = 0$$

$$\bar{u} = -\nabla z \times \nabla \psi$$

IRROTATIONAL

$$\nabla \times \bar{u} = 0$$

$$\bar{u} = \nabla \varphi$$

THEN

$$\nabla \cdot \nabla \varphi = \underline{\underline{\nabla^2 \varphi = 0}}$$

$$\nabla \times (\nabla z \times \nabla \psi) = 0$$

OR $\underline{\underline{\nabla^2 \psi = 0}}$ (z-COMPONENT

$$\bar{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$$

$$\bar{u} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right)$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

VELOCITY POTENTIAL

STREAM FUNCTION (OR VECTOR POTENTIAL)

ARE EQUIVALENT FLOW DESCRIPTIONS

IF ONE IS KNOWN, OTHER IS KNOWN.

Complex Velocity Potential

MANY 2D POTENTIAL FLOW PROBLEMS CAN BE MORE EASILY SOLVED WITH A MATHEMATICAL "TRICK": COMPLEX VARIABLES

LET $W(z) \equiv \varphi + i\psi$

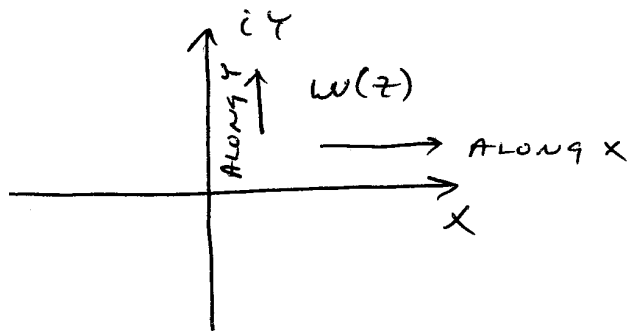
$z \equiv x + iy$

THEN $\text{REAL}(w) = \varphi$, $\text{IMAGINARY}(w) = \psi$

WHY? BECAUSE CAUCHY-RIEMAN CONDITIONS ARE SATISFIED FOR $W(z)$! THIS MEANS DERIVATIVES OF $W(z)$ IN COMPLEX PLANE DO NOT MATTER IN WHICH <u>DIRECTION</u> YOU MAKE THE DERIVATIVES!

E.G. $W(z)$ IS AN ANALYTIC FUNCTION OF z. (EXCEPT AT SINGULARITIES!)

Example Derivatives



$$w(z) = \varphi + i\psi$$

$$\left.\frac{dw}{dz}\right|_{\text{ALONG } X} = \underbrace{\frac{\partial\varphi}{\partial x}}_{u_x} + i\underbrace{\frac{\partial\psi}{\partial x}}_{-iu_y}$$

$$\left.\frac{dw}{dz}\right|_{\text{ALONG } iY} = \underbrace{-i\frac{\partial\varphi}{\partial y}}_{-iu_y} + \underbrace{\frac{\partial\psi}{\partial y}}_{u_x}$$

SO DERIVATIVES ARE EQUAL SINCE $w = \varphi + i\psi$ SATISFY CAUCHY-RIEMANN CONDITIONS!

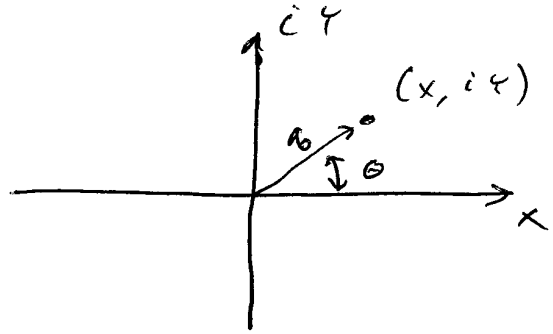
$$\frac{dw}{dz} = u_x - iu_y = \text{COMPLEX VELOCITY OF FLOW}$$

$$\text{REAL}\left(\frac{dw}{dz}\right) = u_x$$

$$\text{IMAGINARY}\left(\frac{dw}{dz}\right) = -u_y$$

Complex Variables

$$z = x + iy = q e^{i\theta} = q(\cos\theta + i\sin\theta)$$



WHEN $\theta = 0, 2\pi, \ldots \quad z = x$

$\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots \quad z = iy$

$\theta = \pi, 3\pi, \ldots \quad z = -x$

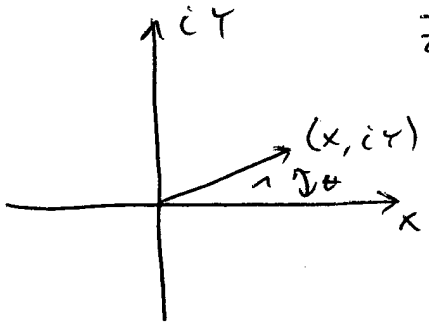
$\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots \quad z = -iy$

ETC

$$z^2 = (x+iy)^2 = q^2 e^{i2\theta} = q^2(\cos 2\theta + i\sin 2\theta)$$

$\theta = 0, \pi, 3\pi \ldots \quad z^2 = x^2$

$\theta = \frac{\pi}{4}, \quad z^2 = iq^2 \; ; \quad \theta = \frac{\pi}{2}, \quad z^2 = (iy)^2 = -y^2$



$$z = x + iy = 1e^{i\theta} = 1(\cos\theta + i\sin\theta)$$

Analytic Function

ANALYTIC FUNCTION, $w(z)$, HAS A CONTINUOUS DERIVATIVE AT z AND THE DERIVATIVE EXISTS TAKEN FROM <u>ANY</u> DIRECTION IN THE z-PLANE

EXAMPLE OF AN <u>ANALYTIC</u> FUNCTION: $w(z) = z$

$$\frac{dw}{dz} = \frac{(z + \Delta x) - z}{\Delta x} = \frac{(z + i\Delta y) - z}{i\Delta y} = 1$$

EXAMPLE OF A <u>NON-ANALYTIC</u> FUNCTION: $w(z) = x - iy$

$$\left.\frac{dw}{dz}\right|_x = \frac{(x + \Delta x - iy) - (x - iy)}{\Delta x} = 1$$

$$\left.\frac{dw}{dz}\right|_y = \frac{(x - iy - i\Delta y) - (x - iy)}{iy} = -1$$

Analytic Functions have Unique Properties

LET $w(z)$ BE ANALYTIC AND $w(z) = u(x,y) + i\,v(x,y)$

THEN

$$\left.\frac{dw}{dz}\right|_{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\left.\frac{dw}{dz}\right|_{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

THEREFORE:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$

CAUCHY-RIEMAN CONDITIONS

(NECESSARY FOR ANY ANALYTIC FUNCTION $w(z)$)

Real and Imaginary Parts of w(z) are Solutions to Laplace's Equation

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

ALSO

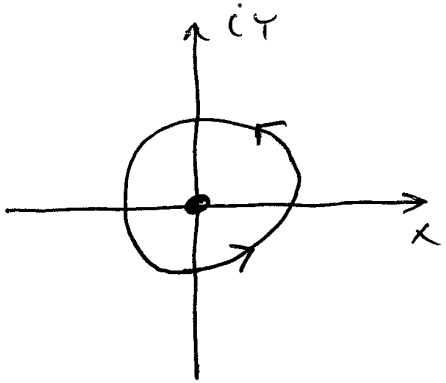
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\nabla^2 u = \nabla^2 v = 0$$ (VERY CONVENIENT!)

Line Integral Around Singularity



$$\oint \frac{dz}{z} = ?$$ POLE AT $z=0$

$$z = r e^{i\theta}$$

$$dz = i r e^{i\theta} d\theta$$ (PATH IS A CIRCLE)

$$\oint \frac{dz}{z} = i \int_0^{2\pi} d\theta = 2\pi i$$

NOTE:

$$\oint z\,dz = i r^2 \int_0^{2\pi} e^{i2\theta} d\theta = 0$$

$$\oint \frac{dz}{z^2} = \frac{i}{r} \int_0^{2\pi} e^{-i\theta} d\theta = 0$$

ETC

$\therefore$ ONLY $\frac{1}{(z-a)}$ TERMS "OR RESIDUE" CONTRIBUTES TO LINE INTEGRAL

Line Integrals do not Depend upon Path

LET

$$F(z_2) - F(z_1) = \oint_{z_1}^{z_2} f(z)\,dz$$

OR $\quad dF = f(z)\,dz$

AND IF DERIVATIVE DOES NOT DEPEND ON DIRECTION, THEN INTEGRAL DOES NOT DEPEND ON PATH.

IF $\quad F(z_2) = F(z_1)$ AS $z_2 = z_1$ THEN

$$\oint f(z)\,dz = 0$$

CALLED <u>CAUCHY'S</u> <u>INTEGRAL</u> <u>THEOREM</u>.

<u>BUT</u> REQUIRES NO "POLES" WITHIN CURVE/PATH

Complex Velocity Potential

WHEN $\nabla \times \bar{u} = \nabla \cdot \bar{u} = 0$ IN 2D, THEN WE CAN DEFINE

φ = VELOCITY POTENTIAL

ψ = STREAM FUNCTION (VECTOR POTENTIAL)

WHERE

$$\nabla^2 \varphi = 0$$

$$\nabla^2 \psi = 0$$

$$\boxed{\begin{aligned} \frac{dw}{dz} &= u_x - i u_y \quad \text{= COMPLEX VELOCITY OF FLOW} \\ \text{REAL}\left(\frac{dw}{dz}\right) &= u_x \\ \text{IMAGINARY}\left(\frac{dw}{dz}\right) &= -u_y \end{aligned}}$$

SO A <u>COMPLEX</u> <u>VELOCITY</u> <u>POTENTIAL</u> IS A CONVENIENT MATHEMATICAL TECHNIQUE ("TRICK") TO SOLVE EULER POTENTIAL FLOW

$$w(z) \equiv \varphi + i\psi$$

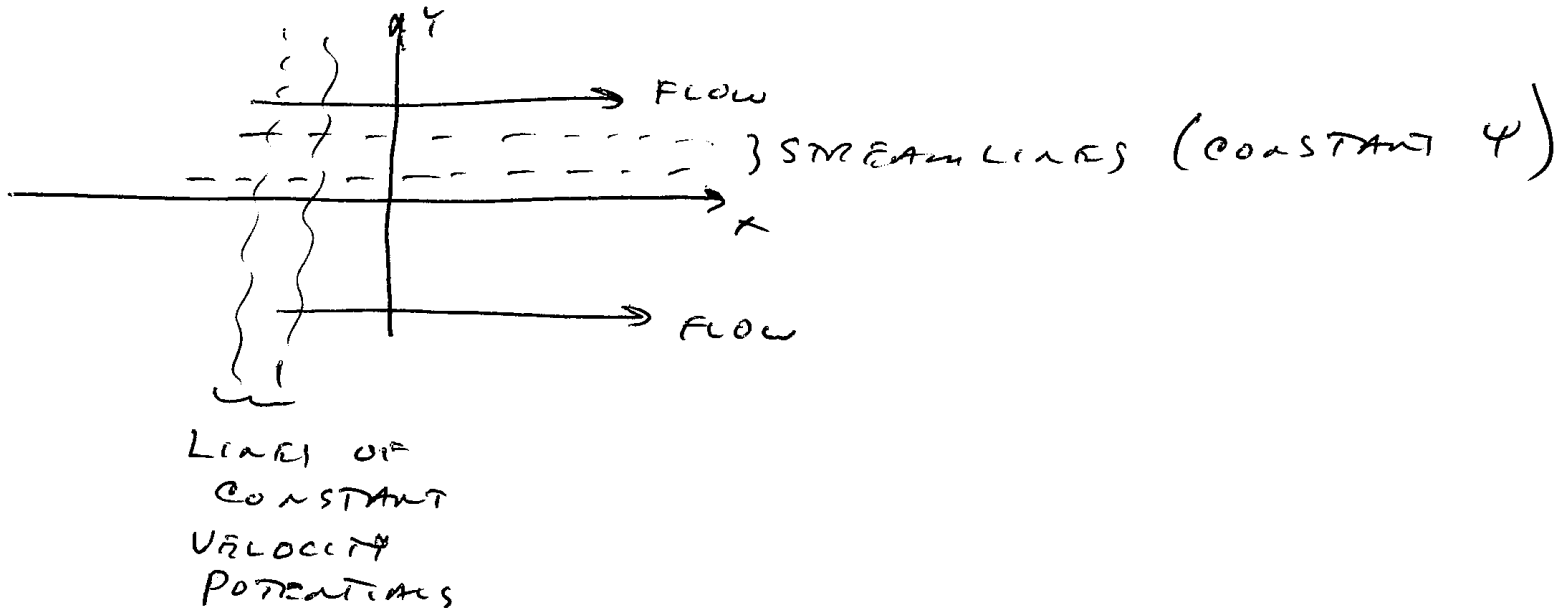
ANY ANALYTIC FUNCTION $w(z)$ IS A SOLUTION FOR 2D POTENTIAL FLOW!!!!

Example Complex Potential

$$w(z) = Az = A(x + iy) \qquad \varphi(x,y) = Ax \qquad \psi(x,y) = Ay$$

$$\frac{dw}{dz} = A = u_x - i\,u_y \implies u_x = A = \text{CONSTANT}$$

$$u_y = 0$$



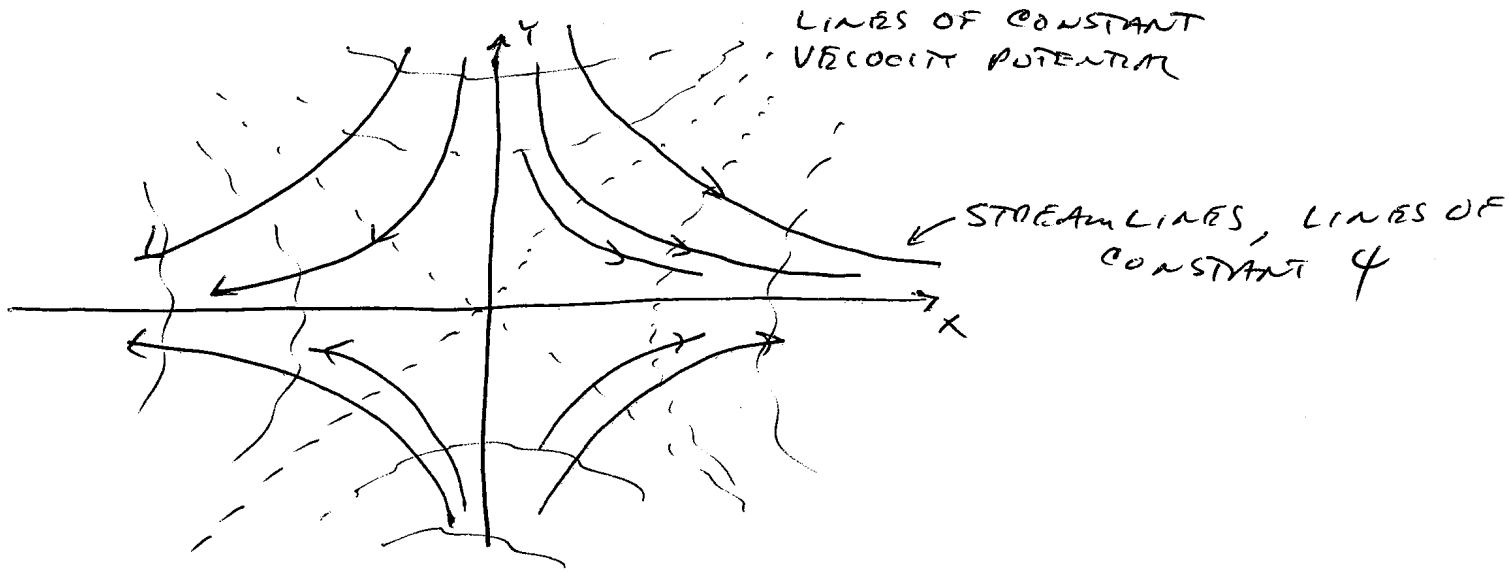
Another Example

$$w(z) = Az^2 = A(x+iy)^2 = A(x^2-y^2) + 2Aixy$$

$$\varphi(x,y) = A(x^2-y^2)$$

$$\psi(x,y) = 2Axy$$

$$\frac{dw}{dz} = 2Az = 2A(x+iy) \Rightarrow u_x = 2Ax \quad , \quad u_y = -2Ay$$



NOTE: $\frac{dw}{dz} \to 0$ AT $z \to 0$. THUS $z = 0 = (x+iy)$ IS A STAGNATION POINT.

Yet Another Example

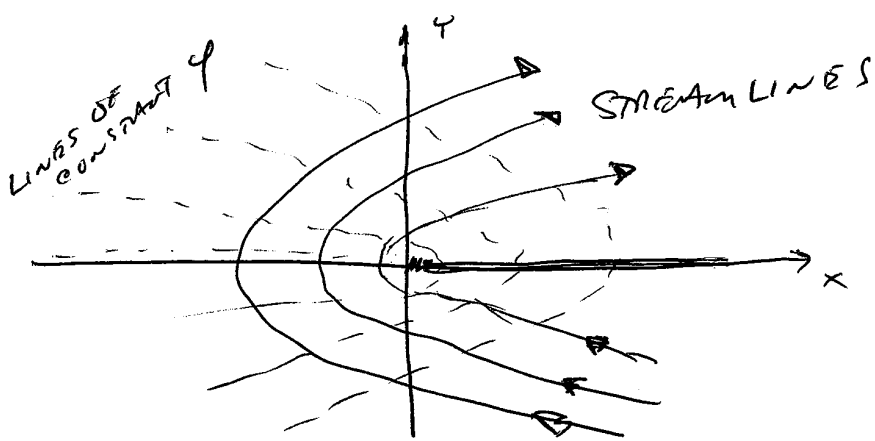
$$w(z) = A\sqrt{z} = A\sqrt{r}\left(\cos\theta/2 + i\sin\theta/2\right)$$ (CYLINDRICAL COORDINATES)

$$\varphi(r,\theta) = A\sqrt{r}\cos\theta/2$$

$$\psi(r,\theta) = A\sqrt{r}\sin\theta/2$$

$$\frac{dw}{dz} = \frac{A}{2}\frac{1}{\sqrt{z}} = \frac{A}{2\sqrt{r}}\left(\cos\theta/2 - i\sin\theta/2\right)$$

$$\Rightarrow \quad u_x = \frac{A}{2\sqrt{r}}\cos\theta/2 \qquad u_y = \frac{A}{2\sqrt{r}}\sin\theta/2$$



Mass Source

$$w(z) = \frac{m}{2\pi} \ln z = \frac{m}{2\pi} \ln(re^{i\theta}) = \frac{m}{2\pi}\left[\ln r + i\theta\right]$$

$$\varphi(r,\theta) = \frac{m}{2\pi} \ln r$$

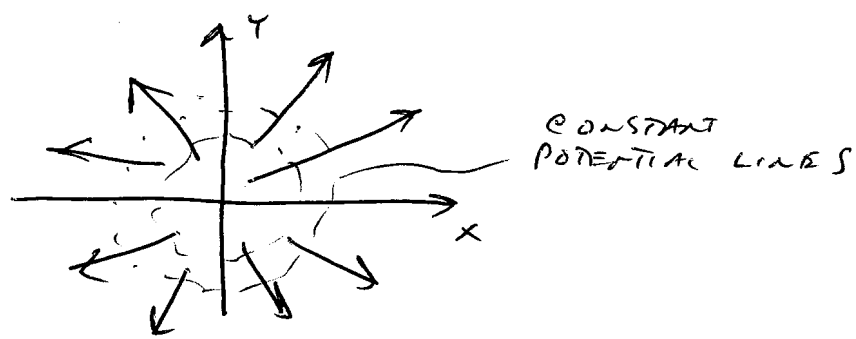
$$\psi(r,\theta) = \frac{m}{2\pi}\theta$$

$$\frac{dw}{dz} = \frac{m}{2\pi z} = \frac{m}{2\pi(x^2+y^2)}(x - iy) \Rightarrow u_x = \frac{m}{2\pi r}\left(\frac{x}{r}\right) \quad u_y = \frac{m}{2\pi r}\left(\frac{y}{r}\right)$$

$$= \frac{m}{2\pi r}\cos\theta \qquad = \frac{m}{2\pi r}\sin\theta$$

So $u_r(r,\theta) = \frac{m}{2\pi r}$

$u_\theta(r,\theta) = 0$



Line Vortex

$$w(z) = -i\frac{\Gamma}{2\pi}\ln z = -i\frac{\Gamma}{2\pi}\left[\ln r + i\theta\right]$$

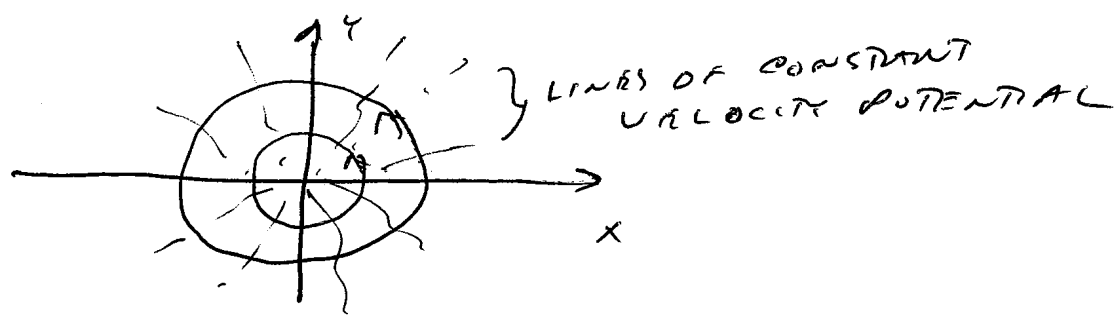
$$\varphi(r,\theta) = \frac{\Gamma}{2\pi}\theta$$

$$\psi(r,\theta) = -\frac{\Gamma}{2\pi}\ln r$$

$$\frac{dw}{dz} = -i\frac{\Gamma}{2\pi z} = -i\frac{\Gamma}{2\pi r^2}(x - iy) = -i\frac{\Gamma}{2\pi r}\left(\frac{x}{r}\right) - \frac{\Gamma}{2\pi r}\left(\frac{y}{r}\right)$$

$$u_x = -\frac{\Gamma}{2\pi r}\sin\theta \qquad u_y = \frac{\Gamma}{2\pi r}\cos\theta$$

on $\quad u_r = 0 \qquad u_\theta = \frac{\Gamma}{2\pi r}$



Summary

- Irrotational, incompressible, 2D flow is especially easy to describe using a "complex" velocity potential.

- Streamlines do not close for irrotational flow.

- The complex potential is defined on the complex z-plane ($z = x + i\,y$) and contains both the velocity potential, $\phi(x,y)$, and the streamfunction, $\psi(x,y)$, or vector potential.