1. Irrotational Flow

2. **Potential Flow in 2D** (incompressible)

3. Complex Variables (quick review)

4. Examples
Irrotational Flow (Ch. 6)

- Potential Flow
- Complex potential (and working with complex representation of 2D flow)
  - Blasius Theorem/Kutta-Zhukhovskiy Theorem
  - Numerical Solution (finite difference)
  - Axisymmetric Flow
Irrotational Flow is Commonplace

*Kelvin's Theorem* states that \( \oint S \vec{n} \cdot d\vec{A} = \oint \vec{u} \cdot d\vec{l} = \Gamma \) is a constant for a surface moving with the fluid.

This is a powerful tool for analyzing flow past objects.

\[
\begin{align*}
\text{At distance} & \quad \nabla \times \vec{u} = 0 \\
\vec{J}_2 &= 0
\end{align*}
\]

Since \( \vec{J}_2 = 0 \) at a distance, \( \vec{J}_2 = 0 \) everywhere along streamlines (pathlines)!

*Note:* the exception occurs when the viscous boundary layer separates at surface.
Irrotational Flow Consequences

- Irrotational flow has no closed streamlines. Since \( \vec{u} \) is always parallel to any pathline \( \int \vec{u} \cdot d\vec{l} = 0 \).

- Since \( \nabla \times \vec{u} = 0 \), then velocity can be expressed as a gradient of a potential \( \vec{u} = \nabla \phi \), \( \nabla \times \nabla \phi = 0 \).

- Euler's equation can be simplified...

\[
\frac{2\vec{u}}{\rho} \cdot \frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} \vec{u}^2 \right) - \vec{u} \times (\nabla \times \vec{u}) = -\nabla p / \rho
\]

\[
\nabla \left( \frac{2\phi}{2t} + \frac{1}{2} \vec{u}^2 + \frac{\rho}{\rho} \right) = 0
\]

Thus, this function is constant/known everywhere in space/time. May be a function of time in space/flow. May be a function of temp.

\[
\frac{2\phi}{2t} + \frac{1}{2} \vec{u}^2 + \frac{p}{\rho} = \text{constant in space} = f(t)
\]
Potential Flow: Incompressible in 2D

Conditions for Incompressible Flow

1. \( |u| \ll \text{Speed of Sound} \)

2. Length \( \ll \text{Speed of Sound} / \text{Frequency} \)

The last condition means that sound propagates across flow much faster than time variation of flow.

\[ \nabla \cdot \bar{u} = 0 \]

MEANS \( \bar{u} = -\nabla z \times \psi \) FOR 2D FLOW

\( z = \text{Symmetry Direction} \)

\( \psi = \text{Stream Function} \)

\[ \nabla \cdot (\nabla z \times \psi) = \nabla \psi \cdot (\nabla \times \nabla z) - \nabla z \cdot (\nabla \times \psi) = 0 \]
Potential Flow in 2D

**INCOMPRESSIBLE**

\[ \nabla \cdot \mathbf{u} = 0 \]
\[ \mathbf{u} = -\nabla \phi \]

Then

\[ \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \]

\[ \mathbf{u} = \left( \frac{2\phi}{2x}, -\frac{2\phi}{2y} \right) \]

\[ \frac{2\phi}{2x} = \frac{2\phi}{2y} \]

**IRROTATIONAL**

\[ \nabla \times \mathbf{u} = 0 \]
\[ \mathbf{u} = \nabla \phi \]

\[ \nabla \times (\nabla \times \phi) = 0 \]

on \nabla^2 \phi = 0 \quad (\phi \text{-component})

\[ \mathbf{u} = \left( \frac{2\phi}{2x}, \frac{2\phi}{2y} \right) \]

\[ \frac{2\phi}{2y} = -\frac{2\phi}{2x} \]

Velocity Potential

Stream Function (or Vector Potential)

An equivalent flow descriptions

If one is known, other is known.
Many 2D potential flow problems can be more easily solved with a mathematical 'trick': complex variables.

Let \( W(z) = \varphi + i \psi \)
\[ z = x + i y \]

Then \( \text{Re}(w) = \varphi \), \( \text{Im}(w) = \psi \)

Why? Because Cauchy-Riemann conditions are satisfied for \( W(z) \)! This means derivatives of \( W(z) \) in complex plane do not matter in which direction you make the derivatives.

E.g. \( W(z) \) is an analytic function of \( z \), (except at singularities!)
Example Derivatives

\[ \frac{dw}{dt} \bigg|_{\text{along } x} = \frac{2\Phi}{2x} + i \frac{2\psi}{2x} \]
\[ w = U_x - i U_y \]
\[ \frac{dw}{dt} \bigg|_{\text{along } y} = -i \frac{2\Phi}{2y} + \frac{2\psi}{2y} \]
\[ w = -i U_y + U_x \]

So derivatives are equal since
\[ w = \varphi + i \psi \text{ satisfy Cauchy-Riemann conditions!} \]

\[ \frac{dw}{dt} = U_x - i U_y = \text{complex velocity of flow} \]
\[ \text{Real} \left( \frac{dw}{dt} \right) = U_x \]
\[ \text{Imaginary} \left( \frac{dw}{dt} \right) = -U_y \]
Complex Variables

\[ z = x + iy = q e^{i \theta} = q (\cos \theta + i \sin \theta) \]

When \( \theta = 0, 2\pi, \ldots \) \( z = x \)

\( \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots \) \( z = iy \)

\( \theta = \pi, 3\pi, \ldots \) \( z = -x \)

\( \theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots \) \( z = -iy \)

RTC

\[ z^2 = (x + iy)^2 = q^2 e^{2i \theta} = q^2 (\cos 2\theta + i \sin 2\theta) \]

\( \theta = 0, \pi, 3\pi \ldots \) \( z^2 = x^2 \)

\( \theta = \frac{\pi}{4}, \) \( z^2 = i q^2 \) \( \theta = \frac{3\pi}{4}, \) \( z^2 = (iy)^2 = -y^2 \)
$$z = x + iy = \lambda e^{i\theta} = \lambda (\cos \theta + i \sin \theta)$$

**Analytic Function**

**Analytic Function**, \(w(z)\), has a continuous derivative at \(z\) and the derivative exists taken from any direction in the \(z\)-plane.

**Example of an analytic function:** \(w(z) = z\)

\[
\frac{dw}{dz} = \frac{(z + \Delta z) - z}{\Delta z} = \frac{(z + i\Delta y) - z}{i\Delta y} = 1
\]

**Example of a non-analytic function:** \(w(z) = x - iy\)

\[
\frac{dw}{dz} \bigg|_x = \frac{(x + \Delta x - iy) - (x - iy)}{\Delta x} = 1
\]

\[
\frac{dw}{dz} \bigg|_y = \frac{(x - iy - i\Delta y) - (x - iy)}{i\Delta y} = -1
\]
Analytic Functions have Unique Properties

Let \( w(z) \) be analytic and \( w(z) = u(x, y) + i v(x, y) \).

Then

\[
\frac{dw}{dz} \bigg|_{dx} = \frac{2u}{2x} + i \frac{2v}{2x}
\]

\[
\frac{dw}{dz} \bigg|_{dy} = -i \frac{2u}{2y} + \frac{2v}{2y}
\]

Therefore:

\[
\frac{2u}{2x} = \frac{2v}{2y}
\]

\[
\frac{2v}{2y} = -\frac{2u}{2x}
\]

Cauchy-Riemann Conditions (necessary for any analytic function \( w(z) \))
Real and Imaginary Parts of \( w(z) \) are Solutions to Laplace's Equation

\[
\begin{align*}
\frac{2u}{2x} &= \frac{2v}{2y} \\
\frac{2u}{2y} &= -\frac{2v}{2x}
\end{align*}
\]

Also

\[
\begin{align*}
\frac{2^2u}{2x^2} &= \frac{2^2v}{2y^2} \\
\frac{2^2u}{2x^2} &= -\frac{2^2v}{2x^2}
\end{align*}
\]

\[\nabla^2 u = \nabla^2 v = 0 \quad \text{(over \& convenient)}\]
Line Integral Around Singularity

\[ \oint \frac{dz}{z} = \pi \text{ at } z = 0 \]

\[ z = \rho e^{i\theta} \]

\[ dz = i\rho e^{i\theta} d\theta \quad \text{(PATH IS A CIRCLE)} \]

\[ \oint \frac{dz}{z} = i \int_{0}^{2\pi} d\theta = 2\pi i \]

**NOTE:**

\[ \oint zdz = i \rho^2 \int_{0}^{2\pi} e^{i2\theta} d\theta = 0 \]

\[ \oint \frac{d\theta}{z^2} = i \int_{0}^{2\pi} e^{-i\theta} d\theta = 0 \]

ETC

\[ \text{ONLY } \frac{1}{z-a} \text{ TERMS "OR RESIDUE" CONtributes TO LINE INTEGRAL} \]
Line Integrals do not Depend upon Path

Let
\[ F(z_2) - F(z_1) = \oint_{z_1} ^ {z_2} f(z) \, dz \]

or \( dF = f(z) \, dz \)

AND IF DERIVATIVE
DOES NOT DEPEND ON
DIRECTION, THEN INTEGRAL
DOES NOT DEPEND ON PATH.

IF \( F(z_2) = F(z_1) \) AS \( z_2 = z_1 \), THEN
\[ \oint f(z) \, dz = 0 \]

Cauchy’s Integral Theorem.

But Requires no
“poles” within curve/path.
Complex Velocity Potential

When \( \forall \times \vec{u} = \nabla \cdot \vec{u} = 0 \) in 2D, then we can define

\[
\psi = \text{velocity potential}
\]

\[
\psi = \text{stream function (vector potential)}
\]

where

\[
\forall^2 \psi = 0
\]

\[
\forall \psi = 0
\]

So a complex velocity potential is a convenient mathematical technique ("trick") to solve Euler potential flow

\[
\psi(z) = \psi + i \psi
\]

Any analytic function \( \psi(z) \) is a solution for 2D potential flow!!!
Example Complex Potential

\[ \omega(z) = A \zeta = A (x + i \gamma) \quad \varphi(x, \gamma) = Ax \quad \psi(x, \gamma) = Ay \]

\[ \frac{dw}{dz} = A = u_x - i u_y \quad \Rightarrow \quad u_x = A = \text{constant} \]

\[ u_y = 0 \]

\[ \text{Streamlines (constant } \psi) \]

\[ \text{Lines of constant velocity potentials} \]
Another Example

\[ \omega(z) = A z^2 = A(x + iy)^2 = A(x^2 - y^2) + 2Aixy \]

\[ \varphi(x, y) = A(x^2 + y^2) \]

\[ \psi(x, y) = 2Axy \]

\[ \frac{d\omega}{dz} = 2Az = 2A(x + iy) \Rightarrow u_x = 2Ax, \ u_y = -2Ay \]

\[ \text{Lines of constant velocity potential} \]

\[ \text{Streamlines, lines of constant } \psi \]

Note: \[ \frac{d\omega}{dz} \to 0 \text{ at } z \to 0. \text{ Thus } z = 0 = (x + iy) \text{ is a stagnation point.} \]
Yet Another Example

\[
\omega(z) = A \sqrt{z} = A \sqrt{\theta} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \quad \text{(differential equation)}
\]

\[
\psi(\gamma, \theta) = A \sqrt{\gamma} \cos \frac{\theta}{2}
\]

\[
\psi(\gamma, \theta) = A \sqrt{\gamma} \sin \frac{\theta}{2}
\]

\[
\frac{\partial \omega}{\partial z} = \frac{A}{2 \sqrt{\gamma}} = \frac{A}{2 \sqrt{\theta}} \left( \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right)
\]

\[
\Rightarrow \quad U_x = \frac{A}{2 \sqrt{\gamma}} \cos \frac{\theta}{2} \quad U_y = \frac{A}{2 \sqrt{\gamma}} \sin \frac{\theta}{2}
\]
Mass Source

\[ \omega (t) = \frac{m}{2\pi} \ln z = \frac{m}{2\pi} \ln (re^{i\theta}) = \frac{m}{2\pi} \left[ \ln r + i \theta \right] \]

\[ \psi (\eta, \theta) = \frac{m}{2\pi} \ln \eta \]

\[ \psi (\eta, \theta) = \frac{m}{2\pi} \theta \]

\[ \frac{d\omega}{dt} = \frac{m}{2\pi} = \frac{m}{2\pi(x^2 + y^2)} (x - iy) \Rightarrow U_x = \frac{M}{2\pi \eta} \left( \frac{x}{\eta} \right) \quad U_y = \frac{M}{2\pi \eta} \left( \frac{y}{\eta} \right) \]

\[ = \frac{M}{2\pi \eta} \cos \theta = \frac{M}{2\pi \eta} \sin \theta \]

So \[ U_\eta (\eta, \theta) = \frac{m}{2\pi \eta} \]
\[ U_\theta (\eta, \theta) = 0 \]

\[
\begin{array}{c}
\text{CONSTANT} \\
\text{POTENTIAL LINES}
\end{array}
\]
Line Vortex

$$\omega(z) = -i \frac{\Gamma}{2\pi} \ln z = -i \frac{\Gamma}{2\pi} \left[ \ln r + i \theta \right]$$

$$\psi(r, \theta) = \frac{\Gamma}{2\pi} \theta$$

$$\psi(r, \theta) = -\frac{\Gamma}{2\pi} \ln r$$

$$\frac{d\omega}{dz} = -i \frac{\Gamma}{2\pi} z = -i \frac{\Gamma}{2\pi} r \left( x - i y \right) = -i \frac{\Gamma}{2\pi r} \left( \frac{x}{r} \right) = \frac{\Gamma}{2\pi r} \left( \frac{x}{r} \right)$$

$$U_x = -\frac{\Gamma}{2\pi r} \sin \theta \quad U_y = \frac{\Gamma}{2\pi r} \cos \theta$$

On $\gamma$

$$U_x = 0 \quad U_\theta = \frac{\Gamma}{2\pi r}$$

Lines of constant velocity potential
Summary

• Irrotational, incompressible, 2D flow is especially easy to describe using a “complex” velocity potential.

• Streamlines do not close for irrotational flow.

• The complex potential is defined on the complex z-plane \( z = x + i\ y \) and contains both the velocity potential, \( \phi(x,y) \), and the streamfunction, \( \psi(x,y) \), or vector potential.