APPH 4200
Physics of Fluids

Introduction

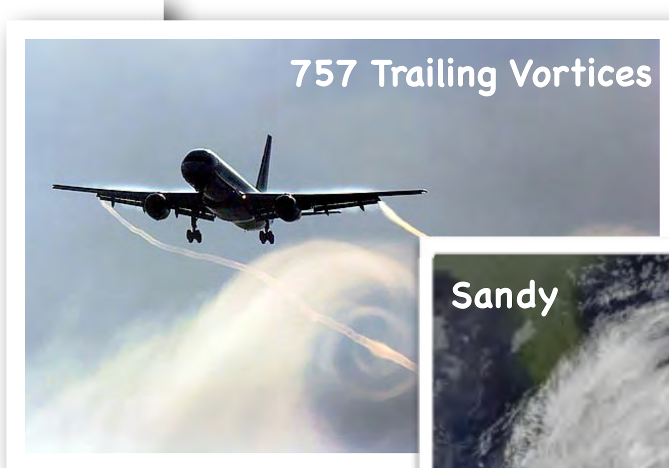
Lecture 1

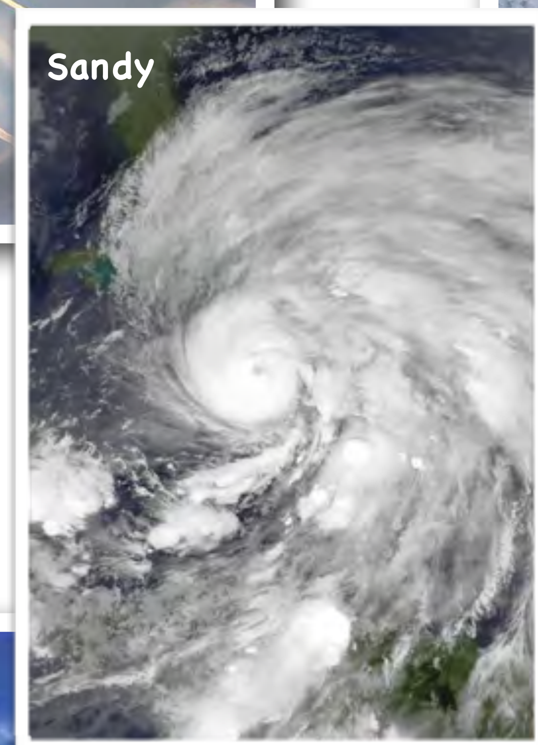
Info

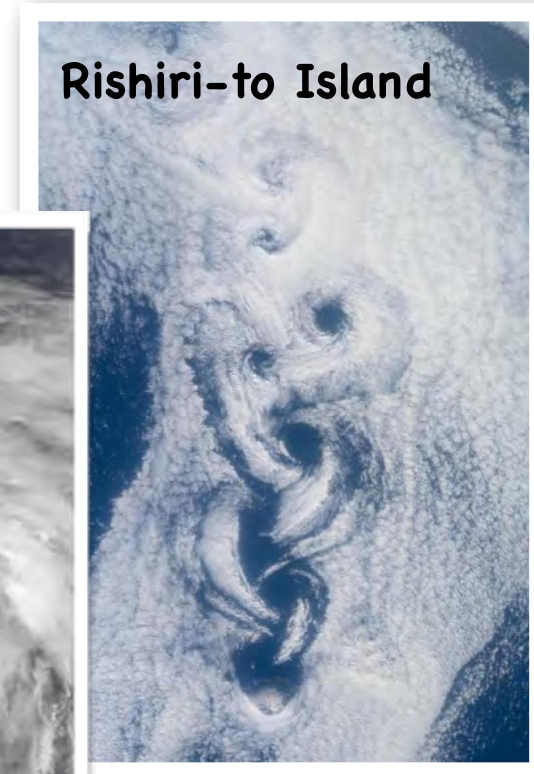
- Course website:
  http://www.apam.columbia.edu/courses/apph4200x/
- Instructor:
  Prof. Mike Mauel, <mauel@columbia.edu>
- TA:
  (N.A.)

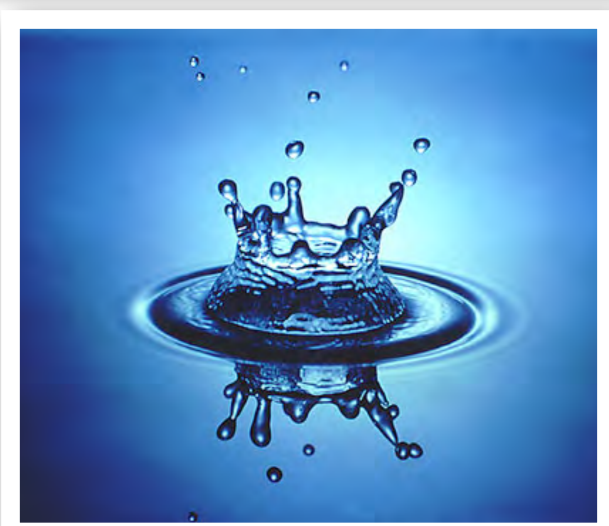
Email us with questions/comments/help requests!

Fluid Behavior is both Complex and Familiar







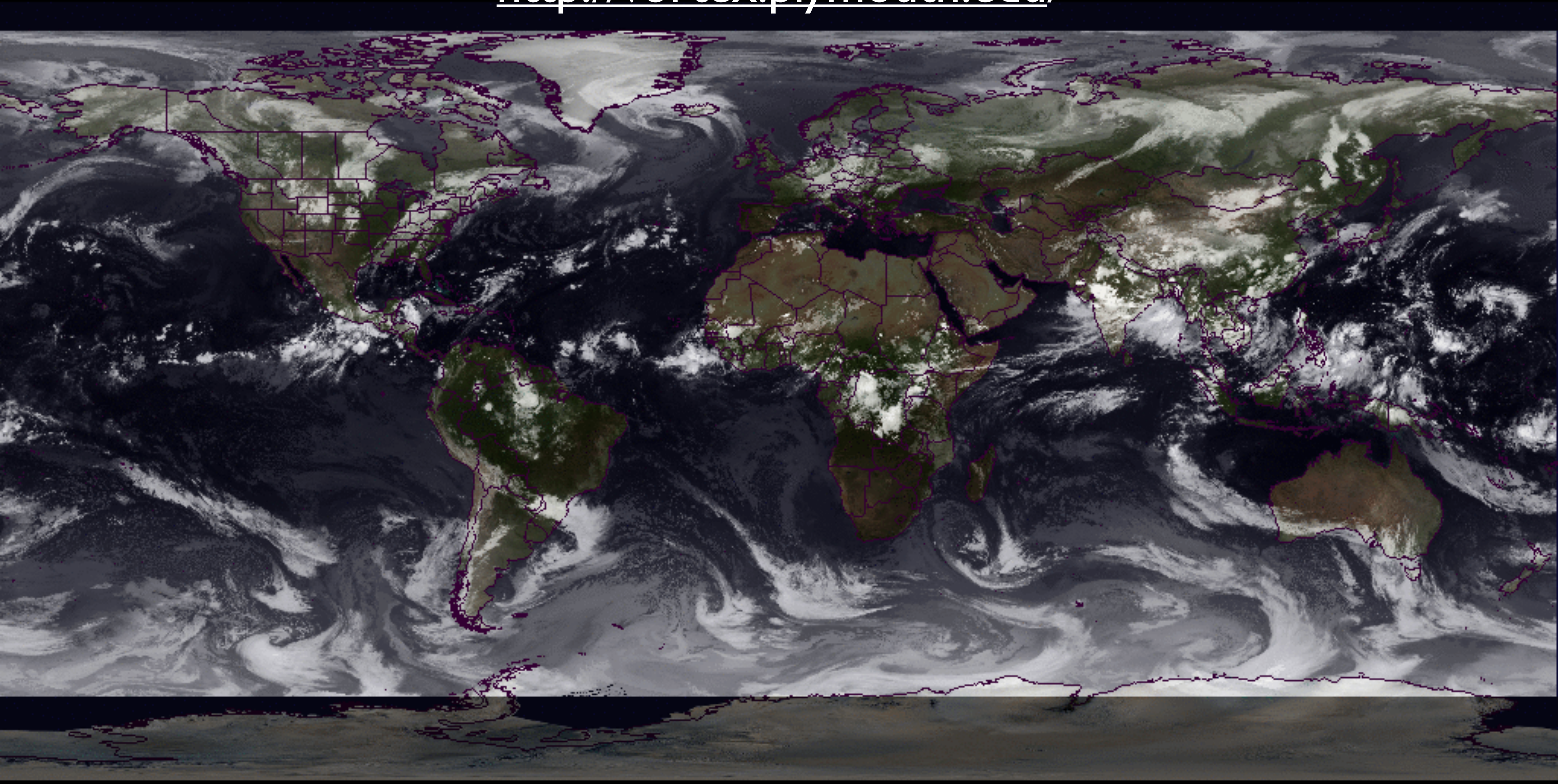


Global IR Satellite

Analysis Valid 12:00Z 31 Aug 2014

Fluid Behavior is both Important and Very Well Measured

http://vortex.plymouth.edu/



Plymouth State Weather Center

Fluid Behavior and Art

The Great Wave off Kanagawa

Starry Night

What is a Fluid?

- Fluid mechanics is continuum mechanics.
- A fluid cannot maintain shear stress without "flowing".
- Includes: liquid, gas, plasma, and mixtures

Fluid Physics History

- **Archmedes** (285-212 B.C.) formulated the laws of buoyancy
- **Leonardo da Vinci** (1452-1519) derived the equation of conservation of mass in one-dimensional steady flow
- **Edme Mariotte** (1620-1684), built the first wind tunnel
- **Isaac Newton** (1642-1727) postulated his laws of motion and the law of viscosity of the linear fluids now called newtonian
- **Euler** developed both the differential equations of motion and their integrated form, now called the Bernoulli equation
- **Lord Rayleigh** (1842-1919) proposed the technique of dimensional analysis
- **Osborne Reynolds** (1842-1912) published the classic pipe experiment in 1883 which showed the importance of the dimensionless Reynolds number
- Viscous-flow theory was available but unexploited since **Navier** (1785-1836) and **Stokes** (1819-1903) had successfully added the newtonian viscous terms to the governing equations of motion. **Unfortunately, the resulting Navier-Stokes equations were too difficult to analyze for arbitrary flows.**

➡ In 1904, a German engineer, **Ludwig Prandtl** (1875-1953), published perhaps **the most important paper ever written on fluid mechanics**. Prandtl pointed out that fluid flows with small viscosity (water and air flows) can be divided into a thin viscous layer, or **boundary layer,** near solid surfaces and interfaces, patched onto a nearly inviscid outer layer, where the Euler and Bernoulli equations apply.

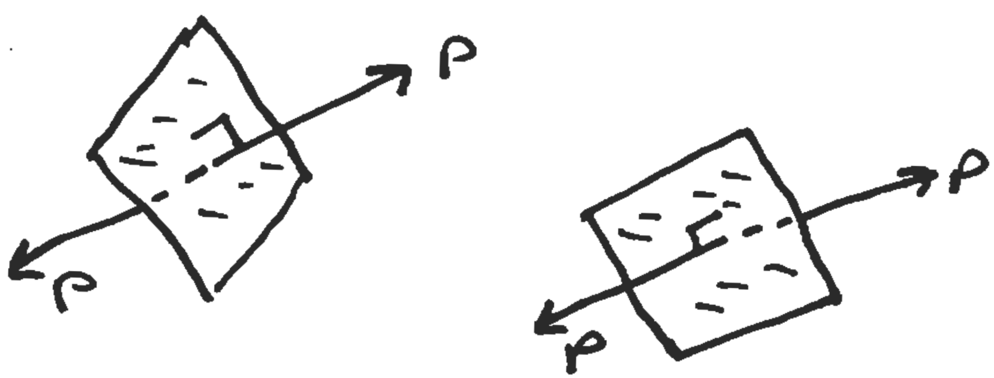
Today's Lecture: Quick Introductions

- Hydrostatics
- Equations of motion and the convective derivative
- Steady flow and Bernoulli's Theorem

Daniel Bernoulli
1700-1782

Hydrostatics

- FLUID AT REST
- FORCES IN BALANCE
- NO SHEAR FORCES
- NORMAL FORCE PER UNIT AREA IS "PRESSURE", EQUAL IN ALL DIRECTIONS! (ISOTROPIC)



Example: Water in a Tank, Lake, Sea, Ocean

ρ = FLUID DENSITY

WATER ≈ INCOMPRESSIBLE

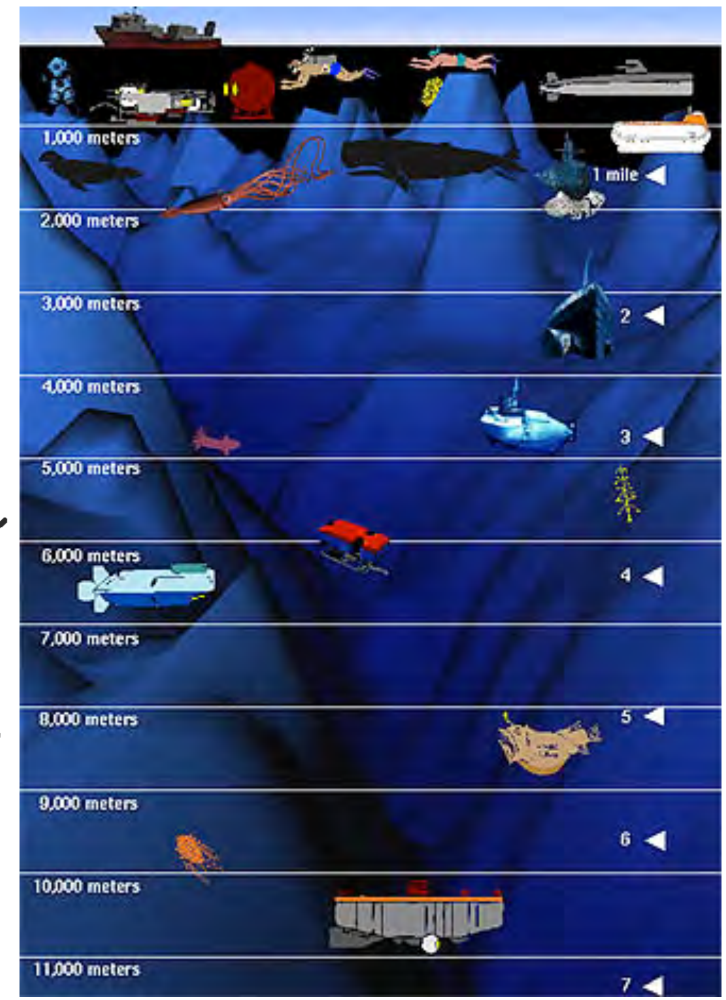
SO $\rho \approx$ CONSTANT

$\approx 1000\ kg/m^3$

$g = 9.8\ m/s^2$ ACCELERATION OF GRAVITY

$-g\rho\hat{z} = -\rho\nabla\varphi$ = GRAVITATIONAL FORCE PER UNIT VOLUME

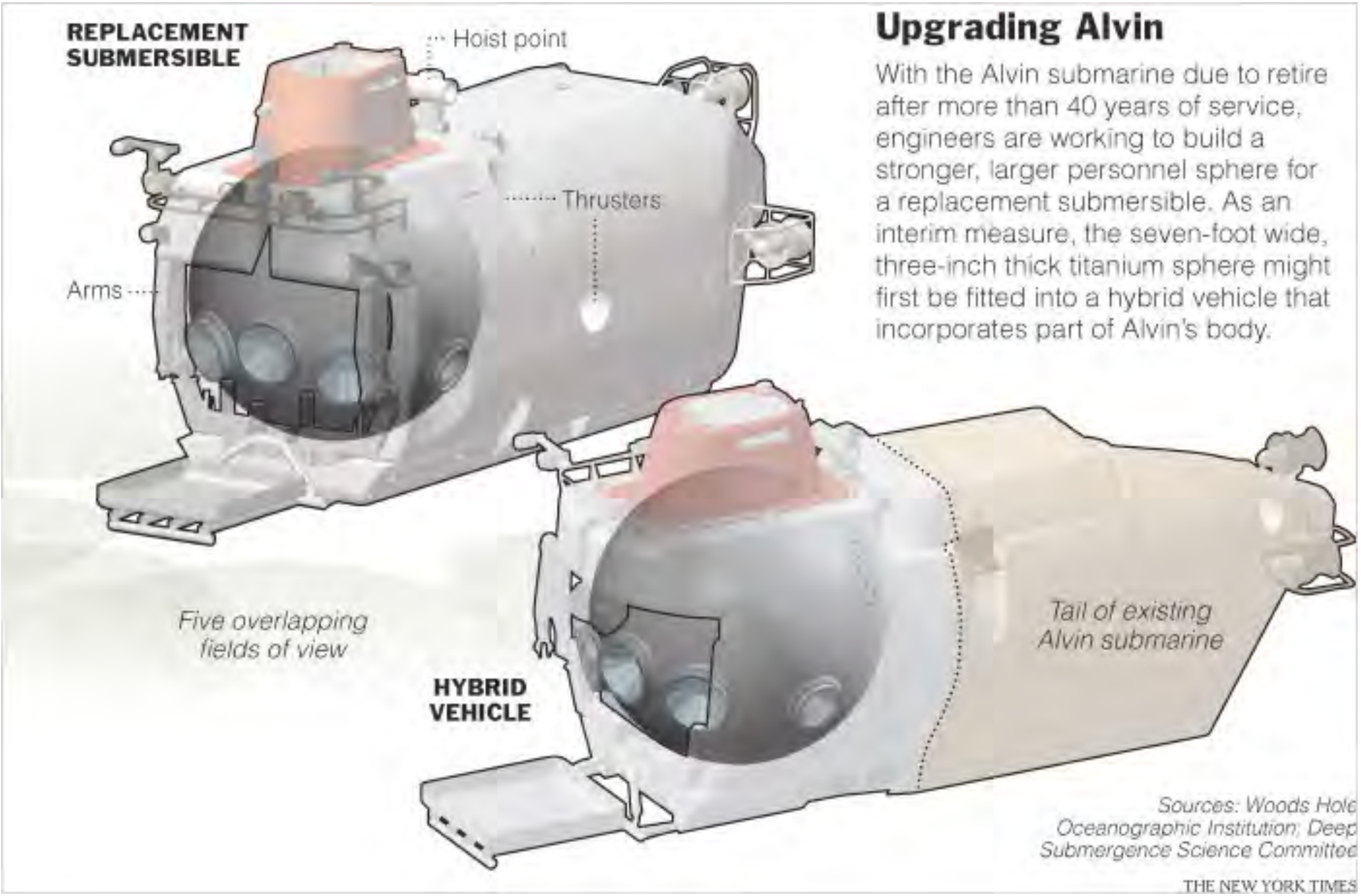
φ = GRAVITIONAL POTENTI



New Sphere in Exploring the Abyss

By WILLIAM J. BROAD
Published: August 25, 2008

The New York Times



Alvin can transport a pilot and two scientists down 2.8 miles, providing access to 62 percent of the dark seabed. **The new vehicle is expected to descend more than four miles**, opening 99 percent of the ocean floor to inquiry. But the greater depth means that the vehicle's personnel sphere and its many other systems will face added tons of crushing pressure.

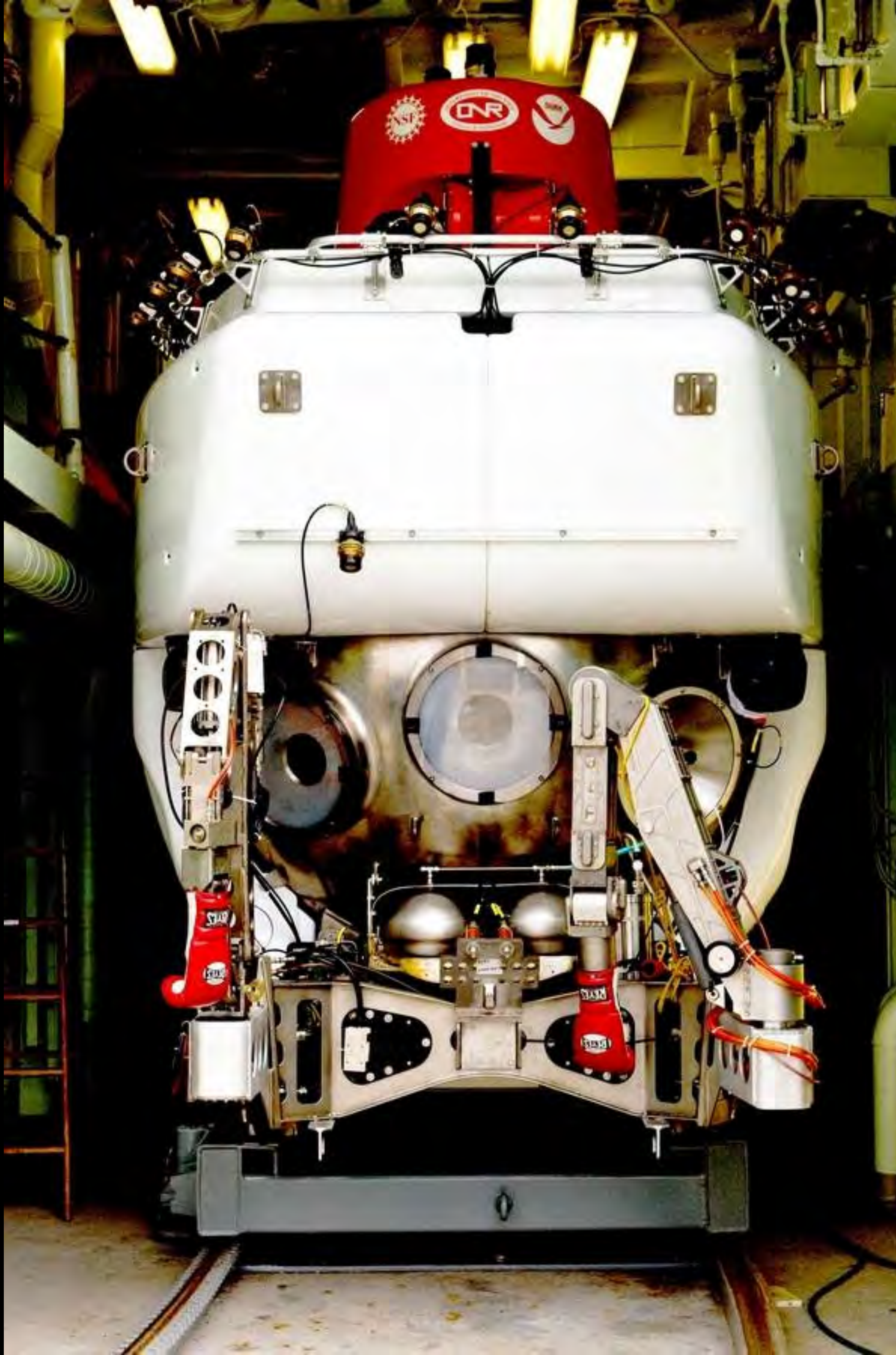
Alvin comes home

Published:
August 29, 2014 11:00AM

Alvin's along for the ride on Atlantis, although it won't be going underwater this time.

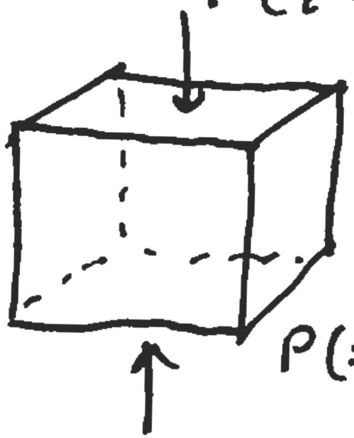
The Alvin, one of the most technologically advanced underwater robots in the world, is kicking back for the current voyage aboard the Atlantis.

But the 50-year-old submersible robot, owned by the U.S. Navy and operated by Woods Hole Oceanographic Institute, is more capable than ever after a three-year, $41 million upgrade.



The $41 million worth of improvements to the submersible Alvin include a new titanium sphere to carry researchers more than 21,000 feet below sea level, five viewing windows and improved buoyancy, lighting and imaging.

What is the net pressure force on a small cube of water?



NET FORCE UPWARD

$$= -\left(P(z+\Delta z) - P(z)\right) dx\, dy$$

BUT $$P(z+\Delta z) \simeq P(z) + \Delta z \frac{\partial P}{\partial z} + \cdots$$

SO

NET UPWARD FORCE $$= -\Delta x \Delta y \Delta z \frac{\partial P}{\partial z}$$

Hydrostatics is the condition of vanishing of the total force

HYDROSTATICS...

$$\left(-\frac{\partial p}{\partial z} - \rho g\right) dx\,dy\,dz = 0$$

OR

$$p(z) = p(0) - \rho g z$$

WITH GRAVITY $= -\nabla\varphi$, THEN $\varphi = gz$

AND

$$p + \rho\phi = \text{CONSTANT}$$

Numbers & Units

NUMBERS

$\rho = 1000 \text{ kg}/m^2$

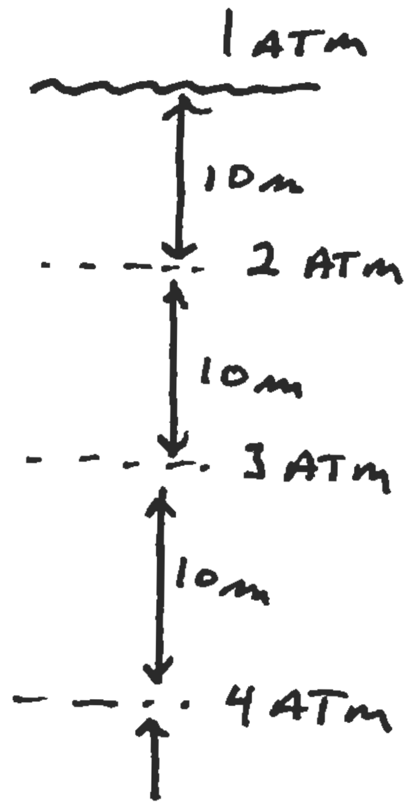
$g = 9.8 \text{ m}/s^2$

$\rho g = 10^4 \text{ kg}/m^2/s^2 = 10^4$ NEWTONS

$= 10^4$ PASCAL

$= 0.1$ bAR (~ 0.1 ATM)

SO 10 METER DEPTH $\sim$ 1 ATM



Hydrostatics in the Atmosphere

PERFECT GAS (AIR) EQUATION OF STATE

$$\rho = \frac{P}{RT}$$

$R = 287\ J/kg/^{\circ}K \sim$ GAS CONSTANT

$T \equiv$ TEMPERATURE

ρ = DENSITY

HYDROSTATIC BALANCE IN ATMOSPHERE WITH CONSTANT TEMPERATURE

$$\frac{\partial P}{\partial z} = -\rho g = -\frac{g}{RT} P$$

SO

$$P(z) = P(0)\, e^{-\left(\frac{g}{RT}\right) z}$$

$$\frac{RT}{g} \sim 7.3\,Km \ (AIR)$$

SCALE HEIGHT

Adiabatic & Isentropic

ADIABATIC MEANS NO HEAT EXCHANGE DURING MIXING

ISENTROPIC MEANS FRICTIONLESS MIXING AS WELL ∴ NO ENTROPY EXCHANGE

THEN $\dfrac{P}{\rho^{\gamma}} \approx$ CONSTANT

WHERE $\gamma \sim \dfrac{c_p}{c_v} \sim 1.4$ FOR AIR

AS A CONSEQUENCE....

$$\left.\begin{aligned} P &= \rho R T \\ P &\propto \rho^{\gamma} \end{aligned}\right\} \quad \frac{T(z)}{T(0)} \sim \left(\frac{P(z)}{P(0)}\right)^{\frac{\gamma-1}{\gamma}}$$

A CONDITION FOR STATIC T AS WELL AS P

Isentropic Atmosphere

$$\ln(T(z)) - \ln T_0 = \frac{\gamma-1}{\gamma}\left[\ln(P(z)) - \ln P_0\right]$$

DIFFERENTIATING...

$$\frac{1}{T}\frac{\partial T}{\partial z} = \left(\frac{\gamma-1}{\gamma}\right)\frac{1}{P}\frac{\partial P}{\partial z}$$

BUT $P = \rho R T$ AND $\frac{dP}{dz} = -g\rho$, SO

$$\frac{1}{T}\frac{\partial T}{\partial z} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{g}{RT} \approx -10^\circ C/km$$

T(z) GENERALLY COOLS WITH HEIGHT FOR A WELL-MIXED ATMOSPHERE.

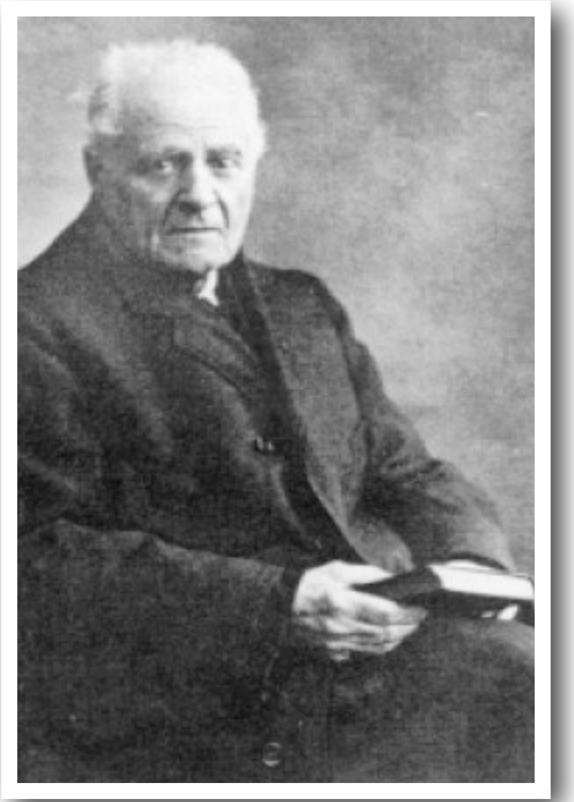
Fluid Equations of Motion

- Fluid element moves with velocity $\bar{u}$
- $\bar{u}$ is a vector $\bar{u} = \{u, v, w\} = \{u_x, u_y, u_z\}$
- Pressure, density, current, temperature, magnetic field, charge... characterize fluid at each location

  ... and evolve in time!
- Often, when $|\bar{u}|$ is less than speed of sound, it is a good assumption to consider fluid dynamics with

  $$\rho \approx \text{constant}$$

This is called the "Boussinesq Approx."

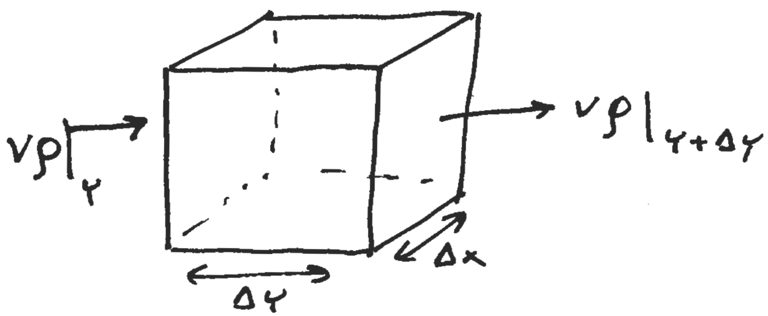


Joseph Boussinesq
1842–1929

Fluid Continuity

(Conservation of Mass)

$\bar{u} = (u, v, w)$



$$\left.\frac{\Delta \rho}{\Delta t}\right|_y \sim -\frac{\left(v\rho|_{y+\Delta y} - v\rho|_y\right)}{\Delta y}$$

$$\sim -\frac{\partial}{\partial y}(v\rho)$$

FOR ALL 3 DIRECTIONS

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \bar{u}) = -\sum_i \frac{\partial}{\partial x_i}(\rho u_i)$$

$$= -\frac{\partial}{\partial x}(\rho u_x) - \frac{\partial}{\partial y}(\rho u_y) - \frac{\partial}{\partial z}(\rho u_z)$$

IF ρ = CONSTANT (BOUSSINESQ) THEN

$\nabla \cdot (\rho \bar{u}) = \nabla \cdot \bar{u} = 0$ "INCOMPRESSIBLE FLOW"

Newton's Law for a Fluid Element

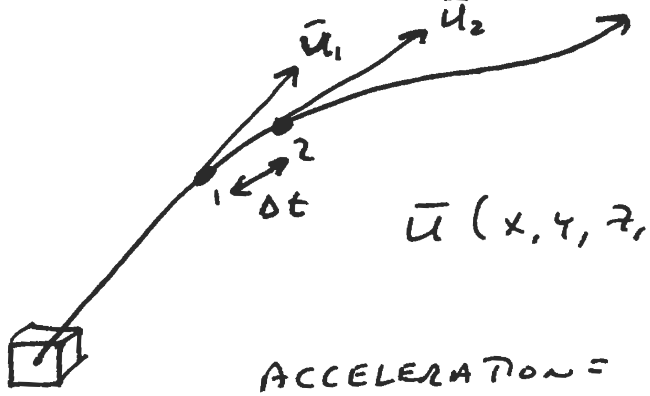
$\rho \times$ (ACCELERATION) = FORCE

$= -\nabla p + \rho \bar{g} + \cdots$

$= -\nabla (p + \rho \varphi) + \cdots$

← GRAVITATIONAL POTENTIAL

WHAT IS ACCELERATION?



$\bar{u}(x, y, z, t)$

ACCELERATION = $\dfrac{\bar{u}_2 - \bar{u}_1}{\Delta t}$ AS $\Delta t \to 0$

$$\bar{u}_2 = \bar{u}(x + u_x dt,\ y + u_y dt,\ z + u_z dt,\ t + \delta t)$$

$$\approx \bar{u}(x, y, z, t) + \frac{\partial \bar{u}}{\partial t}\delta t + \frac{\partial \bar{u}}{\partial x} u_x dt + \frac{\partial \bar{u}}{\partial y} u_y dt + \frac{\partial \bar{u}}{\partial z} u_z dt + \cdots$$

CONVECTIVE OR ADVECTIVE DERIVATIVE

SO

ACCELERATION = $\dfrac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \bar{\nabla})\bar{u}$ AND NEWTON'S LAW IS

$$\boxed{\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \bar{\nabla})\bar{u} = -\frac{1}{\rho}\bar{\nabla} p - \nabla \varphi}$$

Vorticity and Definitions

FROM PROBLEM #3 IN 1ST PROBLEM SET

$$(\bar{u}\cdot\bar{\nabla})\bar{u} = (\nabla\times\bar{u})\times\bar{u} + \frac{1}{2}\bar{\nabla}(\bar{u}\cdot\bar{u})$$

SO THAT

$\bar{\Omega} = \nabla\times\bar{u} \equiv$ VORTICITY

$$\frac{\partial\bar{u}}{\partial t} + \bar{\Omega}\times\bar{u} + \frac{1}{2}\bar{\nabla}u^2 = -\frac{\nabla P}{\rho} - \nabla\varphi$$

IF $\bar{\Omega} = 0$, THEN FLOW IS "IRROTIONAL"

IRROTATIONAL AND INCOMPRESSIBLE FLOWS SATISFY

$$\nabla\cdot\bar{u} = 0 \qquad \nabla\times\bar{u} = 0$$

JUST LIKE EQUATIONS FOR ELECTROSTATICS AND MAGNETOSTATICS IN A CHARGE-FREE OR CURRENT-FREE REGION.

Bernoulli's Principle

FOR STEADY, INCOMPRESSIBLE FLOW...

$$\bar{u} \cdot \left[\frac{\partial \varphi}{\partial t} + \bar{\Omega} \times \bar{u} = -\bar{\nabla}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right)\right]$$

($\frac{\partial \varphi}{\partial t} \to 0$; $\bar{u} \cdot (\bar{\Omega} \times \bar{u}) \to 0$ WHY?)

OR

$$\bar{u} \cdot \bar{\nabla}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = 0$$

BERNOULLI'S LAW STATES THAT

$$\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = \text{CONSTANT ALONG A STREAMLINE}$$

FURTHERMORE, IF $\bar{\Omega} = 0$, THEN

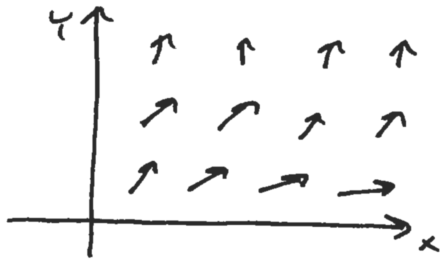
$$\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = \text{CONSTANT EVERYWHERE!}$$

VERY IMPORTANT: AIRPLANE WING LIFT, VERY USEFUL PRESSURE-FLOW RELATIONSHIP!

(CAN BE SHOWN TO BE EQUIVALENT TO CONSERVATION OF ENERGY.)

Visualizing Fluid Flow

- STREAMLINES = INSTANTANEOUS CURVES EVERYWHERE TANGENT TO FLOW



- PATH LINES = TRAJECTORY OF A FLUID ELEMENT AS IT TRAVELS. (TRACER)

PATHLINES ⟺ STREAMLINES IN STEADY FLOW

- STREAKLINES = "SMOKE LINES"

  = THE LOCATION OF ALL OF THE PATH ENDS THAT INITIATED AT A FIXED POINT

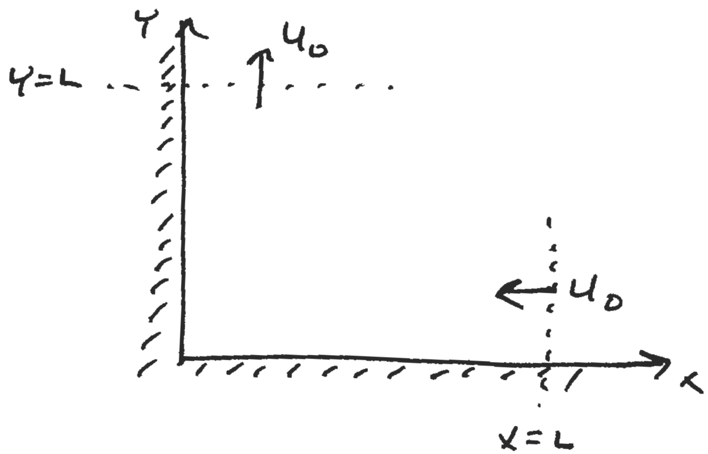
TWO EXAMPLES ....

(1) FLOW AT A CORNER

(2) FLOW AROUND A CYLINDER

$\frac{\partial}{\partial t} \to 0$ STEADY FLOW | 2D $\bar{u} = (u_x, u_y, 0)$ | $\nabla \cdot \bar{u} = 0$, $\nabla \times \bar{u} = 0$

Flow at a Corner



$\nabla \cdot \bar{u} = 0$
$\nabla \times \bar{u} = 0$

(LIKE ELECTRO- OR MAGNETOSTATICS)

TWO SOLUTION METHODS (REALLY EQUIVALENT)

- VELOCITY POTENTIAL

$$\bar{u} = \bar{\nabla} \varphi \quad \text{so} \quad \nabla \times \nabla \varphi = 0$$

$$\nabla \cdot \bar{u} \Rightarrow \boxed{\nabla^2 \varphi = 0}$$ LAPLACES EQUATION

- STREAM FUNCTION

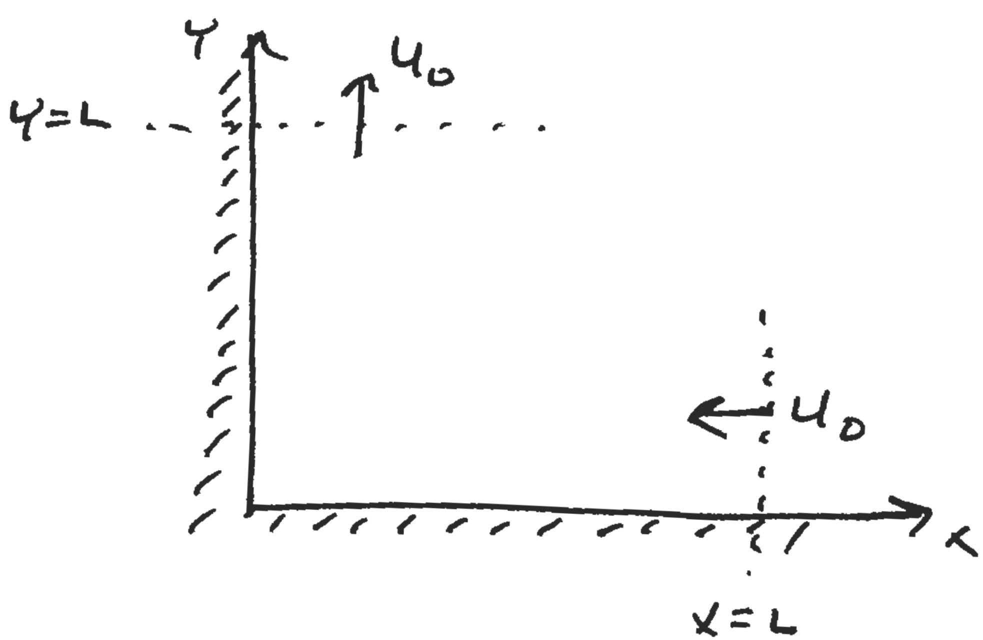
$$\bar{u} = \hat{z} \times \nabla \psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0 \right)$$

$$\nabla \cdot (\hat{z} \times \nabla \psi) = 0$$

$$\nabla \times (\hat{z} \times \nabla \psi) = \boxed{\nabla^2 \psi = 0}$$

Boundary Conditions

- NORMAL FLOW TO WALL MUST VANISH
- FLOW AT $X = L$ ~ $\bar{u} = (-u_0, 0, 0)$
- FLOW AT $Y = L$ ~ $\bar{u} = (0, u_0, 0)$



Solution Using Velocity Potential

(and Streamlines)

SOLUTION USING VELOCITY POTENTIAL...

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$\frac{\partial \varphi}{\partial y} = 0$ AT $y = 0$

$\frac{\partial \varphi}{\partial x} = 0$ AT $x = 0$

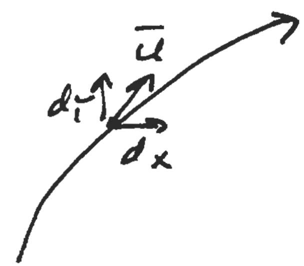
$$\varphi(x, y) = C_1 (x^2 - y^2)$$

$$u_x = 2 C_1 x \qquad u_y = -2 C_1 y$$

BOUNDARY CONDITIONS AT $x, y = L$...

SET $C_1 = -u_0/2L$, SO $\varphi(x, y) = -\frac{u_0}{2L}(x^2 - y^2)$

STREAMLINES



$$\frac{dx}{u_x\, dt} = \frac{dy}{u_y\, dt}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln y = -\ln x + c \text{ OR}$$

$$xy = \text{CONSTANT}$$

Solution Using Streamfunction

$$\bar{U} = \hat{z} \times \nabla \psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0 \right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2} = 0 \quad \text{AND} \quad \frac{\partial \psi}{\partial x} = 0 \text{ AT } y = 0, \quad \frac{\partial \psi}{\partial y} = 0 \text{ AT } x = 0$$

SO $\psi(x,y) = C_1 x y$

$U_x = -U_0$ AT $x = L \Rightarrow C_1 = \frac{U_0}{L}$

$$\boxed{\psi(x,y) = \frac{U_0}{L} x y}$$

$\psi =$ CONSTANT IS A STREAMLINE

NOTE: LINES OF CONSTANT φ, ψ ARE ORTHOGONAL

$\nabla \varphi \cdot \nabla \psi = 0$ ← CAN YOU PROVE THIS?

$\nabla \varphi \cdot (\hat{z} \times \nabla \psi) = u^2$

What is the Pressure Along the Corner?

AT $x = L$, FLUID IS CHARACTERIZED BY

u_0, P_0

USE BERNOULLI'S LAW...

$$\frac{1}{2}\rho u_0^2 + P_0 = \frac{1}{2}\rho u^2 + P(x,y)$$

BUT $u^2 = \bar{u}\cdot\bar{u} = \frac{u_0^2}{L^2}(x^2+y^2)$ AND

ALONG LOWER SIDE ($y=0$)

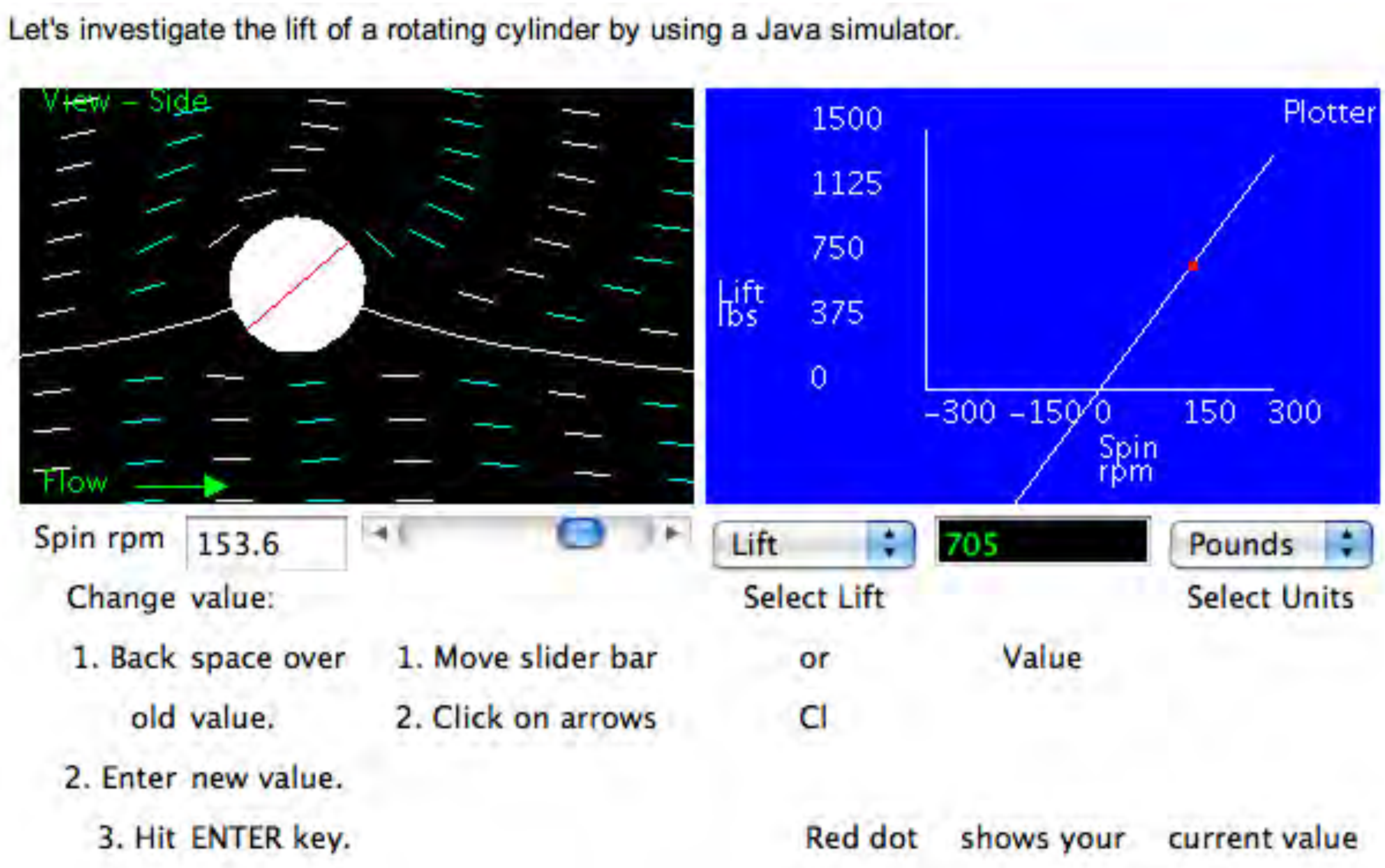
$$P(x,y) = P_0 + \frac{1}{2}\rho\left(u_0^2 - u^2\right)$$

$$= P_0 + \frac{1}{2}\rho u_0^2\left(1 - \frac{x^2}{L^2}\right)$$

$\therefore$ PRESSURE <u>INCREASES</u> INTO CORNER

$$P(0,0) = P_0 + \frac{1}{2}\rho u_0^2$$

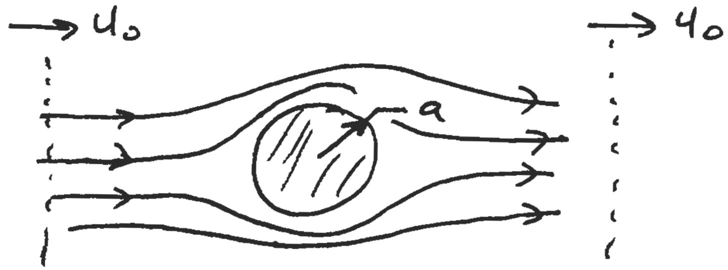
Rotating Cylinder



http://www.grc.nasa.gov/WWW/K-12/airplane/cyl.html

Flow Around a Cylinder

(See Ch. 6-9)



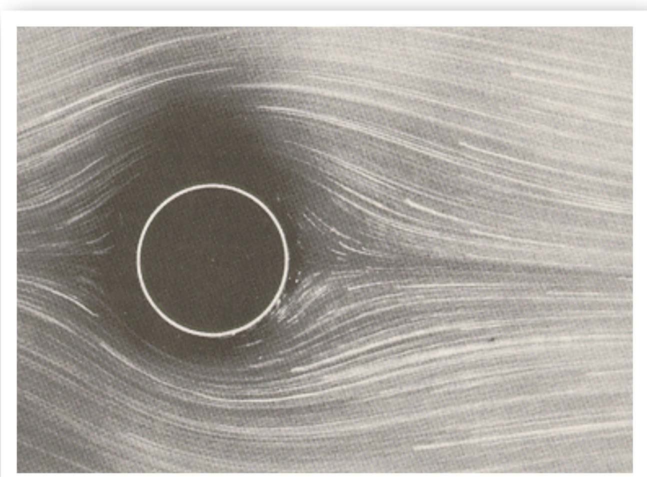
USE VELOCITY POTENTIAL

CYLINDRICAL COORDINATES (SEE APPENDIX B)

$$\nabla^2 \varphi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$$

BOUNDARY CONDITIONS:

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=a} = 0 \qquad u_r = u_0 \cos\theta \text{ AS } r \to \infty$$
$$u_\theta = -u_0 \sin\theta \text{ AS } r \to \infty$$



SOLUTION:

$$\varphi(r,\theta) = f(r)\cos\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) - \frac{f}{r^2} = 0$$

LET $f \propto r^\alpha$ $\quad f' \propto \alpha r^{\alpha-1}$ $\quad f'' \sim \alpha(\alpha-1) r^{\alpha-2}$

SUBSTITUTING

$$\alpha(\alpha-1) + \alpha - 1 = 0 \Rightarrow \underline{\alpha = \pm 1}$$

Velocity Potential Solution

$$\varphi(r,\theta) = \left(C_1 r + \frac{C_2}{r}\right)\cos\theta$$

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = 0 = \left(C_1 - \frac{C_2}{a^2}\right)\cos\theta \Rightarrow \boxed{C_2 = C_1 a^2}$$

As $r \to \infty$

$$u_r = \frac{\partial\varphi}{\partial r} = C_1\cos\theta = u_0\cos\theta \quad \boxed{C_1 = u_0}$$

THUS,

$$\varphi(r,\theta) = u_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$u_r(r,\theta) = u_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

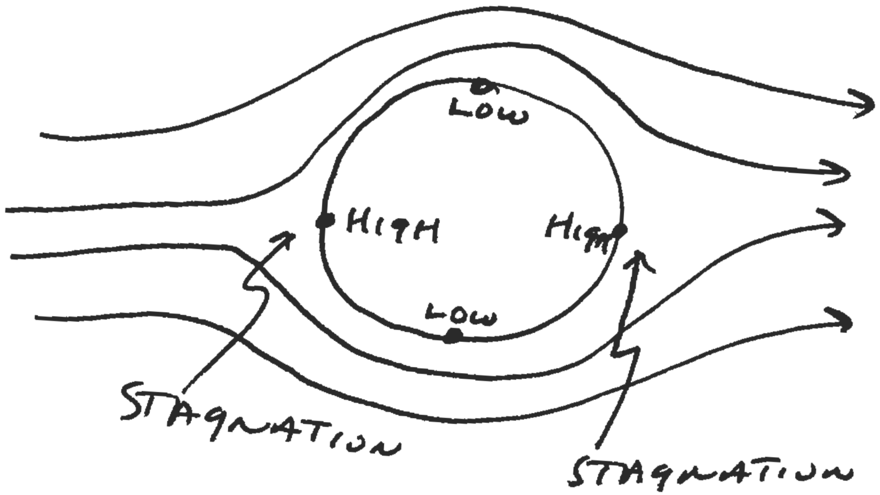
$$u_\theta(r,\theta) = -u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

What is the Pressure at the Surface of the Cylinder?

$$u^2(r=a) = u_\theta^2 = 4u_0^2 \sin^2\theta$$

BERNOULLI'S LAW GIVES

$$p(r=a,\theta) = p_\infty + \frac{1}{2}\rho\left(u_0^2 - 4u_0^2\sin^2\theta\right)$$



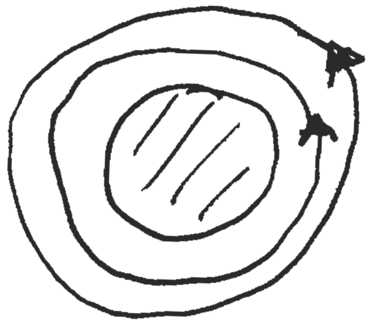
LOWEST PRESSURE AT $\theta = 90°, 270°$

HIGHEST PRESSURE AT $\theta = 0, 180°$

LOW PRESSURE $= p_\infty - 2\rho u_0^2$

HIGH PRESSURE $= p_\infty + \frac{1}{2}\rho u_0^2$

A Solution with Circulation



ROTATING, CIRCULATING FLOW ABOUT THE CYLINDER

AXISYMMETRIC FLOW

$$\bar{u} = (0, u_\theta, 0)$$

$$u_\theta(r) = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} \quad \text{AND} \quad \nabla^2 \phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r}\right)$$

SOLUTION: $u_\theta(r) = \frac{C_1}{r} \qquad \varphi = C_1 \theta$

DEFINE CIRCULATION ABOUT CYLINDER AS

$$\Gamma \equiv \oint_{C_1} \bar{u} \cdot d\bar{s} = \int_0^{2\pi} u_\theta \, a \, d\theta = 2\pi C_1$$

SO $u_\theta(r) = \frac{\Gamma}{2\pi r}$ WHERE Γ = CIRCULATION

Equations for Incompressible and Irrotational Flow are Linear

$\nabla \cdot \bar{u} = 0 \qquad \nabla \times \bar{u} = 0$

SO IF $\bar{u}_1$ AND $\bar{u}_2$ ARE SOLUTIONS THEN ALSO

$\bar{u}_3 = \bar{u}_1 + \bar{u}_2$ IS A SOLUTION.

Add Circulation to Flow

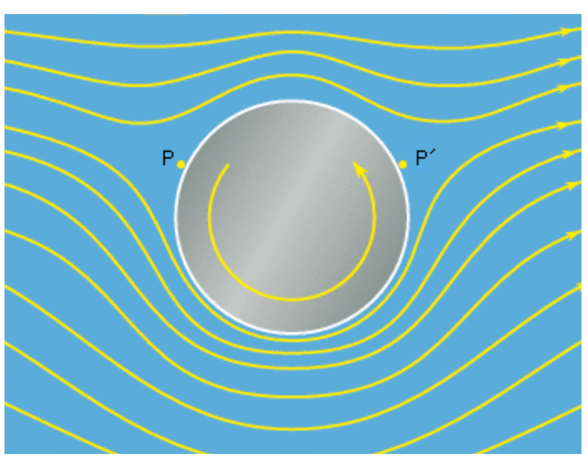
ADD CIRCULATION TO FLOW PAST CYLINDER...,

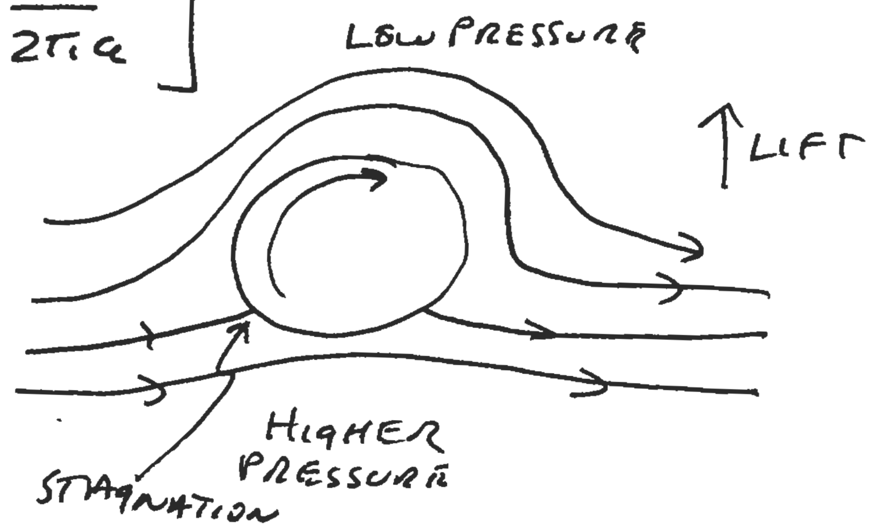
$$u_r(r,\theta) = U_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$u_\theta(r,\theta) = -U_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

BERNOULLI'S LAW AT SURFACE...

$$\frac{1}{2}\rho u^2 = \frac{1}{2}\rho\left[2U_0\sin\theta - \frac{\Gamma}{2\pi a}\right]^2$$





Summary

- Hydrostatics
- Equations of motion and the convective derivative
- Steady flow and Bernoulli's Theorem
- Next Lecture:
    - Vectors and Tensors; Surfaces and Volumes