APPH 4200 PHYSICS OF FLUIDS HW#6 SOLUTIONS

$\boxed{\text{CH 12 P 2}}$ NAVIER-STOKES IN CYLINDRICAL COORDINATES FOR A THIN ROTATING FLUID.

SIMPLIFYING NOTATION:

$$\frac{\partial}{\partial t} \Rightarrow \sigma \qquad \hat{p} = p/\rho_0 \qquad \frac{\partial}{\partial r} \Rightarrow D \qquad \frac{\partial}{\partial z} \Rightarrow ik$$

$$\frac{\partial}{\partial r} + \frac{1}{r} \Rightarrow D' \qquad \frac{\partial^2}{\partial z^2} \Rightarrow -k^2 \qquad DD' = \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}$$

[NOTE: IN MOST CASES, $D \to D'$ SINCE GRADIENTS SCALE LIKE $R_2 - R_1 = d$ AND $d \ll R$.]

N.S.

$\hat{r}$: $\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_r + 2\frac{V}{r}\tilde{u}_\theta = D\hat{p}$

$\hat{\theta}$: $\boxed{\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_\theta - (D'V)\tilde{u}_r = 0}$

$\hat{z}$: $\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_z = ik\hat{p}$ ← STEP 1 ← STEP 2

$\nabla\cdot u$: $D'\tilde{u}_r + ik\tilde{u}_z = 0$

- STEP #1: ELIMINATE $\tilde{u}_z = i D'\tilde{u}_r / k$

$$\frac{\nu}{k^2}\left(DD' - k^2 - \frac{\sigma}{\nu}\right)(D'\tilde{u}_r) = \hat{p}$$

- STEP #2: ELIMINATE $\hat{p}$

$$\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_r + 2\frac{V}{r}\tilde{u}_\theta = \frac{\nu}{k^2}\left(DD' - k^2 - \frac{\sigma}{\nu}\right)(DD'\tilde{u}_r)$$

OR

$$\boxed{\frac{\nu}{k^2}\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\left(DD' - k^2\right)\tilde{u}_r - 2\frac{V}{r}\tilde{u}_\theta = 0}$$

NOW WE HAVE TWO EQUATIONS FOR TWO UNKNOWNS $\tilde{u}_r$ AND $\tilde{u}_\theta$ $\boxed{\text{EQS IN "BOXES"}}$

CH 12 P2 CONT.

IF $d = R_2 - R_1 \ll (R_2, R_1)$ THEN $D' \sim D$

ALSO $V(r) = Ar + B/r$ (SEE TEXT FOR A AND B!)

THEN $(D'V) = 2A$

OUR TWO EQUATIONS BECOME:

$$\left(D^2 - k^2 - \frac{\sigma}{\nu}\right)\left(D^2 - k^2\right)\tilde{u}_r = \frac{2k^2}{\nu}\frac{V(r)}{r}\tilde{u}_\theta$$

$$\left(D^2 - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_\theta = \frac{2A}{\nu}\tilde{u}_r$$

$$= -2\frac{\left(\Omega_1 R_1^2 - \Omega_2 R_2^2\right)}{\left(R_2^2 - R_1^2\right)}\frac{1}{\nu}\tilde{u}_r$$

WHERE $\frac{V(r)}{r} = A + B/r^2$.

- STEP #3: LET'S NORMALIZE VARIABLES AND PARAMETERS

$$\frac{\sigma d^2}{\nu} \to \sigma' \qquad k \to k'/d \qquad D \to \frac{1}{d}\frac{\partial}{\partial x} = \frac{1}{d}D'$$

AND $xd + R_1 = r$ SO THAT

$$A + \frac{B}{r^2} \cong A + \frac{B}{R_1^2} - \frac{2B}{R_1^3}d\,x$$

$$\boxed{A + \frac{B}{R_1^2} = \Omega_1} \quad \text{AND} \quad -\frac{2B}{R_1^3}d \approx \Omega_1\left(\frac{\Omega_2}{\Omega_1} - 1\right)\cancel{\left(\frac{R_2}{R_1}\right)^{2}}$$

WE OBTAIN:

$$\left(D^2 - k^2 - \sigma\right)\left(D^2 - k^2\right)\tilde{u}_r = \frac{2k^2}{\nu}d^2\Omega_1\left(1 + \alpha x\right)\tilde{u}_\theta$$

$$\left(D^2 - k^2 - \sigma\right)\tilde{u}_\theta = -2\frac{\left(\Omega_1 R_1^2 - \Omega_2 R_2^2\right)}{R_2^2 - R_1^2}\frac{d^2}{\nu}\tilde{u}_r$$

IF WE NORMALIZE $\frac{2k^2}{\nu}d^2\Omega_1\tilde{u}_\theta \to \tilde{u}_\theta'$ THEN WE TA

OBTAIN:

$$\left(D^2 - k^2 - \sigma\right)u_\theta' = -4\frac{\left(\Omega_1 R_1^2 - \Omega_2 R_2^2\right)}{R_2^2 - R_1^2}\frac{d^4\Omega_1}{\nu^2}k^2\tilde{u}_r \quad \text{QED}$$

first of the two integrals included in I_4 is positive definite for $\mu > 0$; but the second is complex. However, the real part of I_4 is positive definite for $\mu > 0$; in fact,

$$\mathrm{re}(I_4) = \int_\eta^1 r\phi(r)\left|\frac{dv}{dr}-\frac{v}{r}\right|^2 dr. \tag{185}$$

For, expanding the integrand in (185), we have

$$\int_\eta^1 r\phi(r)\left|\frac{dv}{dr}-\frac{v}{r}\right|^2 dr = \int_\eta^1 \phi(r)\left(r\left|\frac{dv}{dr}\right|^2+\frac{|v|^2}{r}\right) dr - \int_\eta^1 \phi(r)\frac{d|v|^2}{dr}\, dr; \tag{186}$$

but

$$\int_\eta^1 \phi(r)\frac{d|v|^2}{dr}\, dr = (1-\mu)\int_\eta^1 \left(\frac{1}{r^2}-\kappa\right)\frac{d|v|^2}{dr}\, dr = (1-\mu)\int_\eta^1 \frac{1}{r^2}\frac{d|v|^2}{dr}\, dr. \tag{187}$$

Therefore, the right-hand side of (186) is, indeed, the real part of I_4.

Returning to equation (180) and equating the real parts of this equation, we obtain

$$\mathrm{re}(\sigma)(I_1-\mathscr{T}a^2I_3)+I_2-\mathscr{T}a^2[a^2I_3+\mathrm{re}(I_4)] = 0. \tag{188}$$

When $\mu > \eta^2$, $\mathscr{T} < 0$ and the coefficient of $\mathrm{re}(\sigma)$ in equation (188) is positive definite; and so also are the remaining terms in the equation. Therefore,

$$\mathrm{re}(\sigma) < 0 \quad \text{for } \mu > \eta^2, \tag{189}$$

and the flow is stable; this result is entirely to be expected on physical grounds. Nevertheless, it appears to be the only one which can be established by general analytical arguments. In particular, it does not seem that one can deduce the general validity of the principle of the exchange of stabilities for this problem. For example, by equating the imaginary parts of equation (180), we obtain (cf. equation (176))

$$\mathrm{im}(\sigma)(I_1+\mathscr{T}a^2I_3) = -2Ta^2\,\mathrm{im}\int_\eta^1 \frac{v}{r^2}\frac{dv^*}{dr}\, dr, \tag{190}$$

and no general conclusions can be drawn from this equation; when $\mu < 0$, even I_3 is not positive definite!

71. The solution for the case of a narrow gap when the marginal state is stationary

If the gap R_2-R_1 between the two cylinders is small compared to their mean radius $\frac{1}{2}(R_2+R_1)$, we need not (as in § 68 (b)) distinguish

(From S. Chandrasekhar "Hydrodynamic and Hydromagnetic Stability"

between D and D_* in equations (161) and (162); and we can also replace $(A+B/r^2)$ which occurs on the right-hand side of equation (161) by

$$\Omega_1\left[1-(1-\mu)\frac{r-R_1}{R_2-R_1}\right]. \tag{191}$$

In rewriting equations (161) and (162) in the framework of these approximations, it will be convenient to measure radial distances from the surface of the inner cylinder in the unit $d = R_2-R_1$. Thus, letting

$$\zeta = (r-R_1)/d, \quad k = a/d, \quad \text{and} \quad \sigma = pd^2/\nu, \tag{192}$$

we have to consider the equations

$$(D^2-a^2-\sigma)(D^2-a^2)u = \frac{2\Omega_1 d^2}{\nu}a^2[1-(1-\mu)\zeta]v \tag{193}$$

and

$$(D^2-a^2-\sigma)v = \frac{2Ad^2}{\nu}u. \tag{194}$$

By the further transformation

$$u \to \frac{2\Omega_1 d^2a^2}{\nu}u, \tag{195}$$

the equations become

$$(D^2-a^2-\sigma)(D^2-a^2)u = (1+\alpha\zeta)v \tag{196}$$

and

$$(D^2-a^2-\sigma)v = -Ta^2u, \tag{197}$$

where, now,

$$T = -\frac{4A\Omega_1}{\nu^2}d^4 \tag{198}$$

and

$$\alpha = -(1-\mu). \tag{199}$$

Equations (196) and (197) must be considered together with the boundary conditions

$$u = Du = v = 0 \quad \text{for } \zeta = 0 \text{ and } 1. \tag{200}$$

We are primarily interested in the solutions of equations (196) and (197) (subject to the boundary conditions (200)) for various values of a for which the real part of σ is zero. The method described in Chapter III, § 31 in a different connexion is applicable to this problem. Thus, we must obtain solutions for two cases: when σ is zero and the marginal state is stationary and when σ is imaginary and the marginal state is oscillatory. In the latter case for each value of a, $i\sigma$ must be determined by the condition that T is real.† In either case, we must find the minimum of T as a function of a; and depending on which of the two minima is lower, we shall have the onset of instability as a stationary secondary flow or as overstability. Careful experiments on the onset of instability

† It is, of course, possible that under certain circumstances solutions with this property do not exist.

(Oxford Press, 1961)

by Taylor and others have failed to reveal any suggestions of overstability. For this reason, the case $\sigma = 0$ is the only one which has been considered in the literature. However, as no general arguments for the validity of the principle of the exchange of stabilities have been found for this problem, the case of overstability requires investigation. We return to this question in § 72.

When the marginal state is stationary, the equations to be solved are

$$(D^2-a^2)^2u = (1+\alpha\zeta)v \tag{201}$$

and

$$(D^2-a^2)v = -Ta^2u, \tag{202}$$

together with the boundary conditions (200).

(*a*) *The solution of the characteristic value problem for the case* $\sigma = 0$

It can be readily verified that the characteristic value problem presented by equations (200)–(202) is not self-adjoint in the usual sense. For this reason, the method to be described below was patterned after the ones which have been found successful in cases where the problems are self-adjoint. However, Roberts has recently found a variational basis for the method; this is considered in Appendix IV.

The method of solution we shall adopt is the following.

Since v is required to vanish at $\zeta = 0$ and 1, we expand it in a sine series of the form

$$v = \sum_{m=1}^{\infty} C_m \sin m\pi\zeta. \tag{203}$$

Having chosen v in this manner, we next *solve* the equation,

$$(D^2-a^2)^2u = (1+\alpha\zeta)\sum_{m=1}^{\infty} C_m \sin m\pi\zeta, \tag{204}$$

obtained by inserting (203) in (201), and arrange that the solution satisfies the four remaining boundary conditions on u. With u determined in this fashion and v given by (203), equation (202) will lead, as we shall presently see, to a secular equation for T.

The solution of equation (204) is straightforward. The general solution can be written in the form

$$u = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2+a^2)^2}\Big\{A_1^{(m)}\cosh a\zeta+B_1^{(m)}\sinh a\zeta+A_2^{(m)}\zeta\cosh a\zeta+ \\ +B_2^{(m)}\zeta\sinh a\zeta+(1+\alpha\zeta)\sin m\pi\zeta+\frac{4\alpha m\pi}{m^2\pi^2+a^2}\cos m\pi\zeta\Big\}, \tag{205}$$

where the constants of integration $A_1^{(m)}$, $A_2^{(m)}$, $B_1^{(m)}$, and $B_2^{(m)}$ are to be

determined by the boundary conditions $u = Du = 0$ at $\zeta = 0$ and 1. These latter conditions lead to the equations:

$$A_1^{(m)} = -\frac{4m\pi\alpha}{m^2\pi^2+a^2}, \qquad aB_1^{(m)}+A_2^{(m)} = -m\pi,$$

$$A_1^{(m)}\cosh a+B_1^{(m)}\sinh a+A_2^{(m)}\cosh a+B_2^{(m)}\sinh a = (-1)^{m+1}\frac{4m\pi\alpha}{m^2\pi^2+a^2},$$

$$A_1^{(m)}a\sinh a+B_1^{(m)}a\cosh a+A_2^{(m)}(\cosh a+a\sinh a)+ \\ +B_2^{(m)}(\sinh a+a\cosh a) = (-1)^{m+1}(1+\alpha)m\pi. \tag{206}$$

On solving these equations, we find that

$$A_1^{(m)} = -\frac{4\alpha m\pi}{m^2\pi^2+a^2},$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\{a+\beta_m(\sinh a+a\cosh a)-\gamma_m\sinh a\},$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\{\sinh^2 a+\beta_m a(\sinh a+a\cosh a)-\gamma_m a\sinh a\},$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\{(\sinh a\cosh a-a)+\beta_m a^2\sinh a-\gamma_m(a\cosh a-\sinh a)\}, \tag{207}$$

where

$$\Delta = \sinh^2 a-a^2,$$

$$\beta_m = \frac{4\alpha}{m^2\pi^2+a^2}[(-1)^{m+1}+\cosh a],$$

and

$$\gamma_m = (-1)^{m+1}(1+\alpha)+\frac{4\alpha}{m^2\pi^2+a^2}a\sinh a. \tag{208}$$

Now substituting for v and u from equations (203) and (205) in equation (202), we obtain

$$\sum_{n=1}^{\infty} C_n(n^2\pi^2+a^2)\sin n\pi\zeta \\ = Ta^2\sum_{m=1}^{\infty}\frac{C_m}{(m^2\pi^2+a^2)^2}\Big\{A_1^{(m)}\cosh a\zeta+B_1^{(m)}\sinh a\zeta+A_2^{(m)}\zeta\cosh a\zeta+ \\ +B_2^{(m)}\zeta\sinh a\zeta+(1+\alpha\zeta)\sin m\pi\zeta+\frac{4\alpha m\pi}{m^2\pi^2+a^2}\cos m\pi\zeta\Big\}. \tag{209}$$

Multiplying equation (209) by $\sin n\pi\zeta$ and integrating over the range of ζ, we obtain a system of linear homogeneous equations for the constants

$\mathscr{C}_m = C_m/(m^2\pi^2+a^2)^2$; and the requirement that these constants are not all zero leads to the secular equation

$$\left\| \frac{n\pi}{(n^2\pi^2+a^2)}\Big\{[1+(-1)^{n+1}\cosh a]A_1^{(m)}+[(-1)^{n+1}\sinh a]B_1^{(m)}+ \right.$$
$$+(-1)^{n+1}\left[\cosh a-\frac{2a}{n^2\pi^2+a^2}\sinh a\right]A_2^{(m)}+$$
$$+\left[(-1)^{n+1}\sinh a-\frac{2a}{n^2\pi^2+a^2}\{1+(-1)^{n+1}\cosh a\}\right]B_2^{(m)}\Big\}+$$
$$\left. +\alpha X_{nm}+\tfrac{1}{2}\delta_{nm}-\tfrac{1}{2}(n^2\pi^2+a^2)^3\frac{\delta_{nm}}{a^2T}\right\| = 0, \quad (210)$$

$$\text{where } X_{nm} = \begin{cases} 0 \text{ if } m+n \text{ is even and } m \neq n, \\ \tfrac{1}{4} \text{ if } m = n, \\ \dfrac{4nm}{n^2-m^2}\left\{\dfrac{2}{m^2\pi^2+a^2}-\dfrac{1}{\pi^2(n^2-m^2)}\right\} \text{ if } m+n \text{ is odd.} \end{cases} \quad (211)$$

On using the first two equations of (206), equation (210) simplifies to the form

$$\left\| \frac{n\pi}{n^2\pi^2+a^2}\Big\{\frac{4m\pi\alpha}{m^2\pi^2+a^2}[(-1)^{m+n}-1]- \right.$$
$$-\frac{2a}{n^2\pi^2+a^2}[(-1)^{n+1}\{A_2^{(m)}\sinh a+B_2^{(m)}\cosh a\}+B_2^{(m)}]\Big\}+$$
$$\left. +\alpha X_{nm}+\tfrac{1}{2}\delta_{nm}-\tfrac{1}{2}(n^2\pi^2+a^2)^3\frac{\delta_{nm}}{a^2T}\right\| = 0; \quad (212)$$

and on substituting for the constants $A_2^{(m)}$ and $B_2^{(m)}$ their explicit solutions given in (207), we find that equation (212) simplifies greatly and we are left with

$$\left\| \frac{4mn\pi^2\alpha}{(n^2\pi^2+a^2)(m^2\pi^2+a^2)}[(-1)^{m+n}-1]- \right.$$
$$-\frac{2amn\pi^2}{(n^2\pi^2+a^2)^2(\sinh^2 a-a^2)}\Big\{(\sinh a\cosh a-a)[1+(1+\alpha)(-1)^{m+n}]+$$
$$+(\sinh a-a\cosh a)[(-1)^{n+1}+(1+\alpha)(-1)^{m+1}]-$$
$$-\frac{4a\alpha\sinh a}{(m^2\pi^2+a^2)}[\sinh a+a(-1)^{m+1}][(-1)^{m+n}-1]\Big\}+$$
$$\left. +\tfrac{1}{2}\delta_{nm}+\alpha X_{nm}-\tfrac{1}{2}(n^2\pi^2+a^2)^3\frac{\delta_{mn}}{a^2T}\right\| = 0. \quad (213)$$

A first approximation to the solution of equation (213) is obtained by setting the (1, 1)-element of the matrix equal to zero. We find

$$\tfrac{1}{2}(\pi^2+a^2)^3\frac{1}{Ta^2} = \tfrac{1}{4}\alpha+\tfrac{1}{2}-$$
$$-\frac{2a\pi^2(2+\alpha)}{(\pi^2+a^2)^2(\sinh^2 a-a^2)}[(\sinh a\cosh a-a)+(\sinh a-a\cosh a)]. \quad (214)$$

On further simplification, this gives

$$T = \frac{2}{2+\alpha}\,\frac{(\pi^2+a^2)^3}{a^2\{1-16a\pi^2\cosh^2\tfrac{1}{2}a/[(\pi^2+a^2)^2(\sinh a+a)]\}}. \quad (215)$$

We observe that, apart from the factor $2/(2+\alpha)$, this expression for T is identical with what was found in Chapter II (§ 17, equation (311)) for the Rayleigh number for the simple Bénard problem by the variational method in the first approximation for the case of two rigid boundaries. Consequently, in this approximation (cf. equation (199)),

$$T_c = \frac{2}{2+\alpha}\times 1715 = \frac{3430}{1+\mu} \quad \text{and} \quad a_{\min} = 3{\cdot}12. \quad (216)$$

We shall see below (§ (*b*)) that for $0 < \mu < 1$ equation (216) gives values for the critical Taylor number which do not differ from those obtained in the higher approximations by more than one per cent. The reason for this relatively high accuracy of the solution in the first approximation will be made apparent in § (*d*).

(*b*) *Numerical results*

A method of solving the infinite order characteristic equation which (213) provides for T would be to set the determinant formed by the first n rows and columns of the secular matrix equal to zero and let n take increasingly larger values. In practice, the usefulness of this method will depend largely on how rapidly the lowest positive root of the resulting equation of order n tends to its limit as $n \to \infty$. It appears that for the problem on hand, the process converges quite rapidly.

In Table XXXII the values of T obtained with the aid of equation (213) in the different approximations are listed for those values of a (for the assigned μ's) at which it was found (by trial and error) that T attained its minimum value. From an examination of this table, it would appear that for $\mu > -1{\cdot}0$, the third approximation provides T to well within one per cent of the true value. For $-3{\cdot}0 \leqslant \mu \leqslant -1{\cdot}0$, the calculations were carried out to as high approximations as seemed

Ch 12 P3 THIS PROBLEM ASKS TO REPEAT THE ANALYSIS OF CHANDRASEKHAR AND DETERMINE THE CRITICAL WAVELENGTH FOR THE ONSET OF CENTRIFUGAL INSTABILITY.

CHANDRASEKHAR'S ANALYSIS IS ATTACHED.

S. CHANDRASEKHAR WAS A PLASMA ASTROPHYSICIST, AND HE WON THE 1983 NOBEL PRIZE FOR PHYSICS FOR HIS WORK ON STELLAR EVOLUTION.

Ch 12 P5

(i) THE RAYLEIGH EQUATION IS LINEAR IN THE STREAM FUNCTION $\bar{U} = \hat{z} \times \nabla \varphi$. SINCE $U_x = \partial\varphi/\partial y$ AND $U_y = -\partial\varphi/\partial x = -jk\varphi$ THEN U_y AND φ MUST SATISFY THE SAME EQUATION. U_y AND φ ALSO SATISFY THE SAME BOUNDARIES SINCE $\tilde{U}_y$ MUST VANISH AT WALLS.

(ii) IF c IS AN EIGENVALUE, THEN

$$(U_0 - c)(\tilde{U}_y'' - k^2 \tilde{U}_y) - U_0'' \tilde{U}_y = 0$$

WHERE $\tilde{U}_y$ IS A COMPLEX PHASOR. THE COMPLEX CONJUGATE OF THE ABOVE EQ IS:

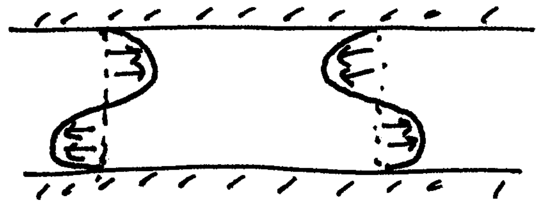
$$(U_0 - c^*)(\tilde{U}_y^{*\prime\prime} - k^2 \tilde{U}_y^*) - U_0'' \tilde{U}_y^* = 0$$

BUT $\tilde{U}_y^*$ SATISFIES THE SAME EQUATION AND BOUNDARY CONDITIONS. THIS MEANS THAT $c^* = c_r - i c_i$ MUST ALSO BE AN EIGENVALUE.

THIS PROPERTY OCCURS BECAUSE THE EQUATION (RAYLEIGH'S) DOES NOT DEPEND UPON $i = \sqrt{-1}$.

Ch 12 P 5 CONT — ANTI-SYMMETRIC JET

(iii) FOR AN ANTI-SYMMETRIC JET, WE HAVE



$U(y) = -U(-y)$ $\qquad$ $U(y)$ $\qquad$ $U(-y)$

$$\frac{\partial U}{\partial y} \to -\frac{\partial U}{\partial y}$$

$$\frac{\partial^2 U}{\partial y^2} \to -\frac{\partial^2 U}{\partial y^2}$$

THEN, TRANSFORMING THE EQUATION TO $y \to -y$, WE HAVE

$$(-U_0 - c)(\tilde{u}_y'' - k^2 \tilde{u}_y) + U_0'' \tilde{u}_y = 0$$

OR $$(U_0 + c)(\tilde{u}_y'' - k^2 \tilde{u}_y) - U_0'' \tilde{u}_y = 0$$

THUS c AND $-c$ MUST BOTH BE EIGENVALUES. THEREFORE, SOLUTIONS MUST EXIST THAT PROPAGATE IN BOTH DIRECTIONS.

(iv) HERE, WE FOLLOW THE "HINT". FOR A SYMMETRIC JET, $\tilde{u}_y(-y)$ AND $\tilde{u}_y(y)$ SATISFY THE SAME EQUATION. THUS, WE CAN DEFINE:

SYMMETRIC SOLUTION: $S(y) \equiv \tilde{u}_y(y) + \tilde{u}_y(-y)$

ANTI-SYMMETRIC SOLUTION: $A(y) \equiv \tilde{u}_y(y) - \tilde{u}_y(-y)$

COMBINING

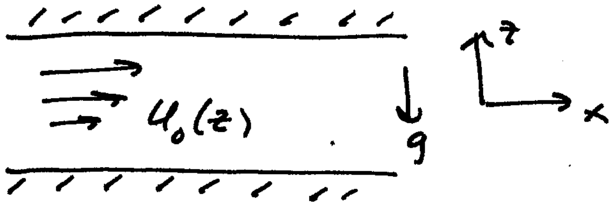
$$A(y) \times [S - EQN] - S(y) \times [A - EQN] = 0$$

RESULTS IN

$$(U_0 - c)(AS'' - SA'') = 0$$

OR $\frac{d}{dy}(AS' - SA') = 0$. THIS GIVES $S \propto A$ WHICH IS IMPOSSIBLE. ONLY ONE OR OTHER.

$\boxed{\text{Ch 12 P.6}}$



THIS QUESTION ASKS US TO REPRODUCE EQ. 12.83 EXCEPT ADDING THE BUOYANCY TERM FROM GRAVITY AND DENSITY STRATIFICATION.

USE NAVIER-STOKES (EULER) AND MASS CONSERVATION:

$$\#1: \quad \frac{\partial \bar{u}}{\partial t} + u_0 \frac{\partial \bar{u}}{\partial x} + \hat{x}\,\tilde{u}_z \frac{\partial u_0}{\partial z} + \frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0} g \hat{z} = 0$$

$$\#2: \quad \frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + u_0 \frac{\partial \tilde{\rho}}{\partial z} + \tilde{u}_z \frac{\partial \rho_0}{\partial z} = 0$$

BUT $\omega_B^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$ = BUOYANCY FREQUENCY^2

- COMBINING:

$$\tilde{u}\cdot(\#1) \Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}\tilde{u}^2\right) + u_0 \frac{\partial}{\partial x}\left(\tfrac{1}{2}\tilde{u}^2\right) + \tilde{u}_x \tilde{u}_z \frac{\partial u_0}{\partial z}$$
$$+ \frac{1}{\rho_0}\tilde{u}\cdot\nabla p + \frac{\rho}{\rho_0} g \tilde{u}_z = 0$$

$$\frac{g^2 \rho}{\rho_0^2 \omega_B^2} \times (\#2) \Rightarrow \frac{g^2}{\rho_0^2 \omega_B^2}\left[\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho^2\right) + \rho u_0 \frac{\partial \tilde{\rho}}{\partial x}\right] - \tilde{u}_z \frac{g\rho}{\rho_0} = 0$$

- ADD

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\frac{g^2\rho^2}{\rho_0^2\omega_B^2}\right) + u_0 \frac{\partial}{\partial x}\left(\tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\frac{g^2\rho^2}{\rho_0^2\omega_B^2}\right)$$
$$+ \underbrace{\frac{1}{\rho_0}\tilde{u}\cdot\nabla p}_{\frac{1}{\rho_0}\left[\nabla\cdot(\tilde{u}p) - p\nabla\cdot\tilde{u}\right]} + \tilde{u}_x \tilde{u}_z \frac{\partial u_0}{\partial z} = 0$$

- FINALLY INTEGRATE OVER CONTROL VOLUME AS ON P. 502
USE BOUNDARY CONDITIONS TO ELIMINATE MOST TERMS
AND THIS RESULTS IN GLOBALLY-INTEGRATED
ENERGY EQUATION!

Ch 13 P1

$$S(\omega) = \frac{1}{2\pi}\int e^{-j\omega t} R(t)\,dt$$

$$= \frac{1}{2\pi}\int (\cos\omega t - i\sin\omega t)\, R(t)\,dt$$

IF $R(t) = R(-t)$, THEN THE TERM WITH $\sin\omega t$ VANISHES.

IF $R(t)$ IS ALSO REAL, THEN $S(\omega)$ IS REAL AND SYMMETRIC.

Ch 13 P2

$$u(t) = U_0 \cos t + \bar{u} \quad (t \to \omega t)$$

$$\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} dt\, u(t) = \bar{u}$$

$$\overline{u^2} = \frac{1}{2\pi}\int_0^{2\pi} dt\left(U_0^2\cos^2 t + 2U_0\bar{u}\cos t + \bar{u}^2\right)$$

$$= \frac{1}{2}U_0^2 + \bar{u}^2$$

$$u_{RMS} = \sqrt{\overline{u^2}} = \sqrt{\tfrac{1}{2}U_0^2 + \bar{u}^2}$$

$$u_{STD}^2 = \overline{(u-\bar{u})^2} = \frac{1}{2\pi}\int_0^{2\pi} dt\, U_0^2\cos^2 t$$

$$= \frac{1}{2}U_0^2$$

SO $u_{STD} = U_0/\sqrt{2}$

Ch 13 P3

$$R(\tau) = \overline{u(t)\,u(t+\tau)}$$

$$= \frac{u^2}{2\pi}\int_0^{2\pi} \cos t' \cos(t+\tau)\,dt'$$

BUT $\cos(t)\cos(t+\tau) = \cos(t)\left[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)\right]$

AND $R(t) = \frac{u^2}{2}\cos(\tau)$ ALSO PERIODIC

Ch 13 P5

HEAT FLUX $= \rho C_p \overline{\tilde{w}\tilde{T}} = \rho C_p \frac{1}{2}\tilde{w}_{RMS}\tilde{T}_{RMS}$

AIR $20°C \Rightarrow$ $C_p \approx 1012$ J/kg°K $\rho = 1.2$ kg/m^3 SO HEAT FLUX $= 61$ W/m^2

Ch 13 P10

UNDERSTANDING TRANSPORT CAUSED BY TURBULENCE IS A CHALLENGE. WHEN THE FLUCTUATIONS CAUSE RANDOM MOTION, THE STATISTICS ARE GAUSSIAN AND RELATIVELY EASY TO DESCRIBE.–

$$\frac{d}{dt}\overline{x^2} = 2\,\overline{x\frac{dx}{dt}} \qquad \text{BUT } x = \int_0^t dt'\,u(t')$$

$$= 2\,\overline{u^2}\int_0^t R(\tau)\,dt$$

AT LONG TIMES, THE AVERAGE OF X FROM IT'S ORIGIN AT $t=0$ SCALES LIKE

$$(\Delta x)^2 \sim \left(\overline{u}^2\, 2\, \mathcal{T}_e\right) t$$

WHERE $\mathcal{T}_e \equiv \int_0^\infty d\tau\, e^{-t^2/\tau_c^2} = \frac{\sqrt{\pi}}{2}\,\tau_c$

CORRELATION TIME

THE PARTICLE DIFFUSIVITY SCALES LIKE:

$$K_p \sim \frac{1}{2}\frac{d}{dt}(\Delta x)^2 \sim u_{rms}^2\,\mathcal{T}_c$$

WITH $t_c = 1s$ $\quad u_{rms} = 1m/s$ $\quad K_p \sim 0.89\ m^2/s$

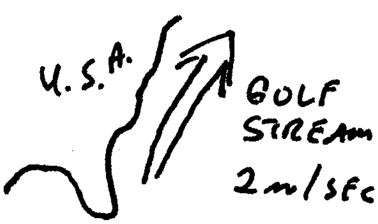
Ch 14 P1

GEOSTROPHIC BALANCE

$f \sim 2\omega \sin(45°N)$

$\sim 10^{-4}\ sec^{-1}$

$g = 9.8\ m/s^2$

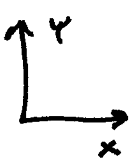


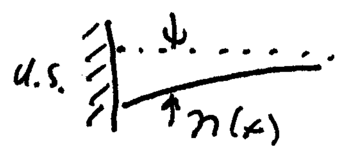
$$-f u_y = \frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

WITH $P = \rho g (H + \eta - z)$

$$\frac{\partial P}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{f u_y}{g} = 2\times 10^{-5}$$

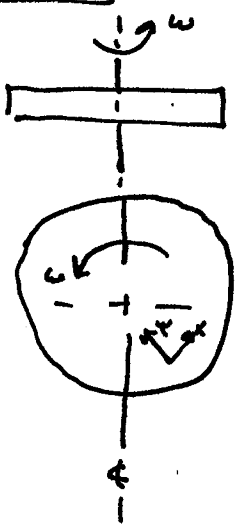
$\approx 2\ m/km$





CORIOLIS FORCE DEPRESSES WATER LEVEL.

Ch 14 P2



δ_E = EKMAN LAYER

$$= \sqrt{\frac{2\nu}{f}}$$

WITHIN VISCOUS EKMAN LAYER, CORIOLIS FORCE IS BALANCED BY VISCOSITY

$$\hat{x}: \quad -f u_y = \nu \frac{\partial^2 u_x}{\partial z^2}$$

$$\hat{y}: \quad f u_x = \nu \frac{\partial^2 u_y}{\partial z^2}$$

THESE EQS ARE SOLVED IN SEC. 6 IN TEXT

SCALE OF LENGTH MUST BE $\propto \sqrt{\nu/f}$

FOR THIS PROBLEM, $\nu = 10^{-6}$ $\quad \omega = 2\pi/6 \quad f = 2\omega$

SO $\quad \delta_E \sim \sqrt{\frac{2\, 10^{-6}\, 6}{2\pi}} = 3\times 10^{-3}\,m$ (3 mm) VERY THIN

Ch 14 P3

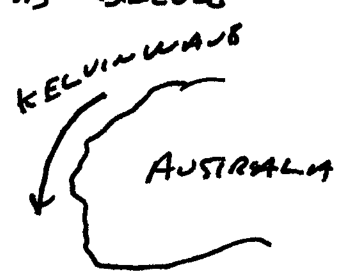
WITHIN AN EKMAN LAYER IS A POLAR FLOW $\quad \int_0^\delta u_y\, dz \sim \frac{1}{2} u \delta$

HERE, $u \sim 10\, m/s$, AND $\delta = \sqrt{2\nu/f} = \sqrt{\frac{2 \cdot 10}{10^{-4}}} \sim 440\,m$

SO $\frac{1}{2} u\delta \sim 2\times 10^{3}\, m^2/sec$, A SIGNIFICANT FLUX.

Ch 14 P5

AS EXPLAINED ON P. 616 IN TEXT, KELVIN WAVES PROPAGATE WITH COAST ON THE LEFT IN SOUTHERN HEMISPHERE. THUS, WAVES PROPAGATE AS BELOW



WITH $H \sim 50\, m$ FOR THERMOCLINE

$\rho \sim 1000\, kg/m^3$

$$C = \sqrt{\underbrace{g\frac{\Delta\rho}{\rho}}_{\text{REDUCED } g \text{ IN CONTINUOUSLY STRATIFIED}} H} = 1\, m/sec$$

WITH $f \sim 2\omega \sin 30^\circ$

$$\Lambda = C/f \sim 14\, km$$

Ch 14 P 7

(i)

$H \sim 10\,\text{Km}$ $\Big\}\ \Delta T \sim 50^\circ C$ $\quad \theta \sim 45^\circ N$

$$\omega_B^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \qquad \text{BUT} \quad \rho = \rho_0\left(1 - \alpha(T - T_0)\right)$$

$$= g\alpha\frac{\Delta T}{H} \qquad \frac{\partial \rho}{\partial z} \sim -\alpha\rho_0\frac{\partial T}{\partial z} \propto \frac{\Delta T}{H}$$

$$\simeq 10^{-4}$$

SO $\omega_B \sim 10^{-2}$ RAD/SEC $\qquad \frac{2\pi}{\omega} = 518\,\text{SEC}$ (8 min)

(ii) FROM P. 608 IN TEXT

$$C \sim \frac{\omega_B H}{\pi} \sim 41\,\text{m/s}$$

(iii) $C_{\text{ROSSBY}} \sim -\beta\frac{C^2}{S^2}$ (FOR $k \sim \frac{1}{R}$)

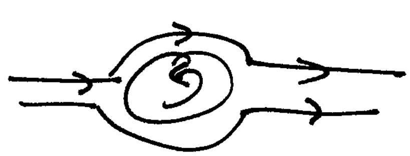
$\beta \sim \frac{2\pi\,\omega\cos\theta}{R} \sim 2\times 10^{-11}$ SO $C_{\text{ROSSBY}} \sim 3.4$ m/SEC

Ch 14 P.8

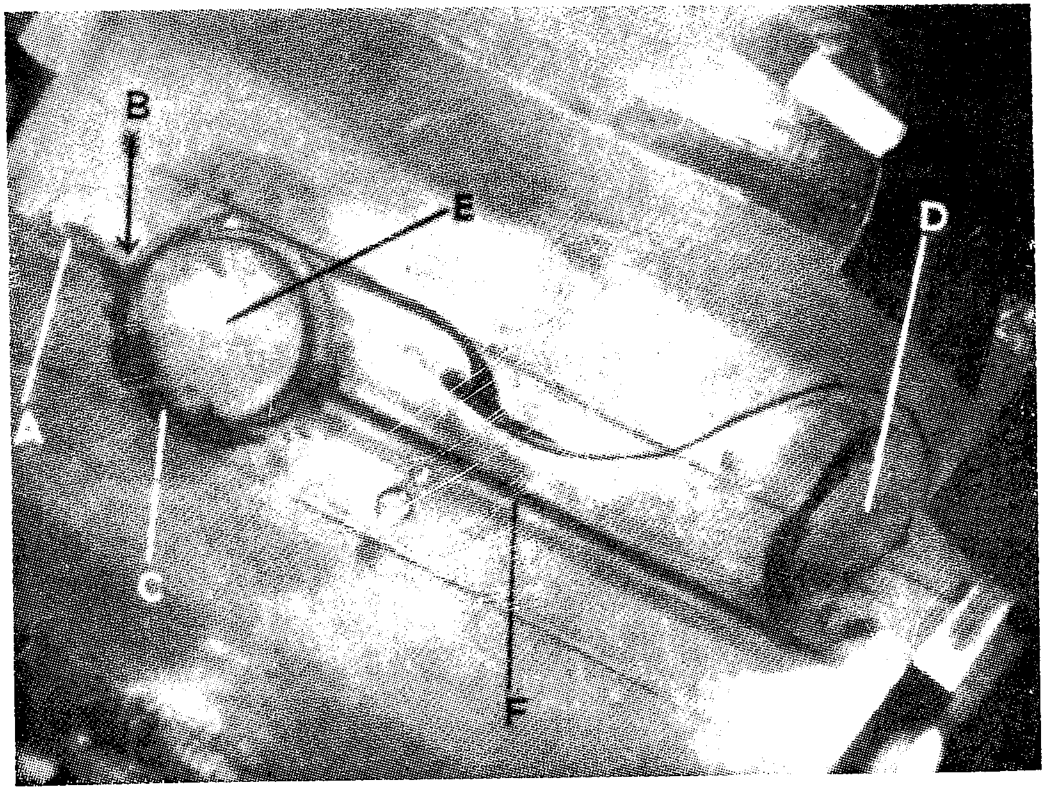
THIS BEAUTIFUL PROBLEM RELATES TO AN EXPERIMENT CONDUCTED BY G.I. TAYLOR IN 1923. ALSO TO TAYLOR-PROUDMAN COLUMNS DESCRIBED IN SEC. 4 OF Ch. 14.

WHEN THE HYDROSTATICS ARE DOMINATED BY THE CORIOLIS FORCE, THEN THE HORIZONTAL FLOW IS INDEPENDENT OF Z.

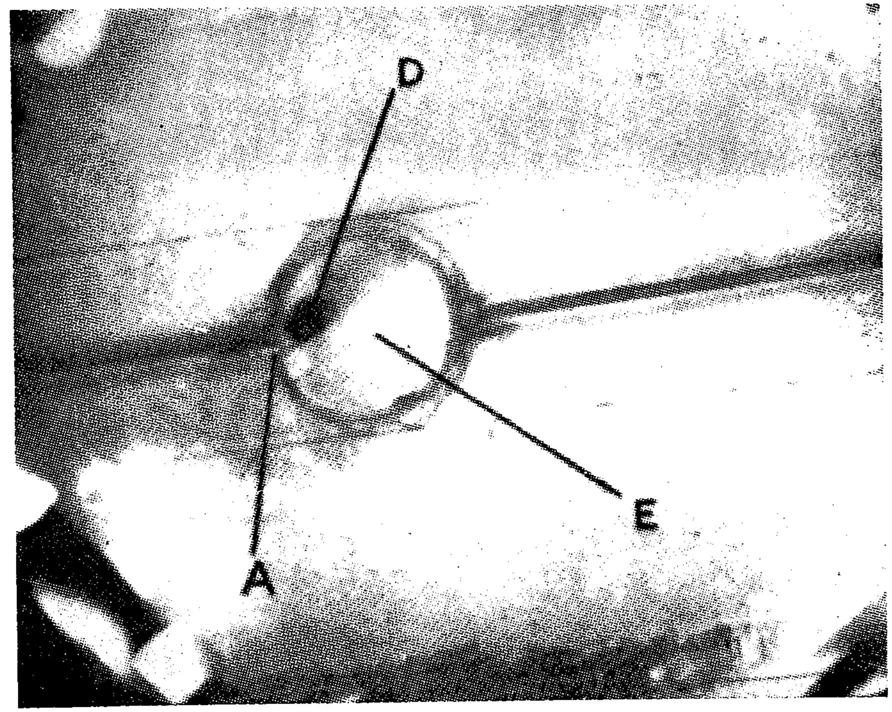
THIS IS EQ. 14-21 CALLED THE TAYLOR-PROUDMAN THEOREM! IN THIS CASE, THE FLOW, u, IS DIVERTED AROUND THE ROTATING CYLINDER AS IF IT ACTUALLY EXTENDS UPWARD!



SEE DISCUSSION AND PHOTOS OF AN ACTUAL EXPERIMENT ATTACHED!



(*a*)



(*b*)

Figure 7.6.3. Motion in a rotating dish of water 4 in. deep due to slow translation of a circular cylinder (E) of height 1 in. from right to left across the bottom of the dish, viewed from above. In (*a*) the dye has been released at a (moving) point A above the top of the cylinder and directly ahead of it, and B (a dividing point), C and D are subsequent positions of the dye; in (*b*) the point of release (A) is within the upward projection of the cylinder and the dye remains in the blob D. The flow evidently has a two-dimensional character. (From Taylor 1923.)

FROM "AN INTRODUCTION TO FLUID DYNAMICS"
BY G.K. BATCHELOR CAMBRIDGE PRESS 1967