APPH 4200 PHYSICS OF FLUIDS HW#6 SOLUTIONS

CH 12 P 2

NAVIER-STOKES IN CYLINDRICAL COORDINATES FOR A THIN ROTATING FLUID.

SIMPLIFYING NOTATION:

$$\frac{\partial}{\partial t} \Rightarrow \sigma \qquad \hat{p} = p/\rho_0 \qquad \frac{\partial}{\partial r} \Rightarrow D \qquad \frac{\partial}{\partial z} \Rightarrow ik$$

$$\frac{\partial}{\partial r} + \frac{1}{r} \Rightarrow D' \qquad \frac{\partial^2}{\partial z^2} \Rightarrow -k^2 \qquad DD' = \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}$$

[NOTE: IN MOST CASES, $D \to D'$ SINCE GRADIENTS SCALE LIKE $R_2 - R_1 = d$ AND $d \ll R$.]

N.S.

$\hat{r}$: $$\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_r + 2\frac{U}{r}\tilde{u}_\theta = D\hat{p}$$

$\hat{\theta}$: $$\boxed{\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_\theta - (D'U)\tilde{u}_r = 0}$$

$\hat{z}$: $$\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_z = ik\hat{p}$$ ← STEP 1

$\nabla\cdot u$: $$D'\tilde{u}_r + ik\tilde{u}_z = 0$$

← STEP 2

- STEP #1: ELIMINATE $\tilde{u}_z = i D'\tilde{u}_r / k$

$$\frac{\nu}{k^2}\left(DD' - k^2 - \frac{\sigma}{\nu}\right)(D'\tilde{u}_r) = \hat{p}$$

- STEP #2: ELIMINATE $\hat{p}$

$$\nu\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\tilde{u}_r + 2\frac{U}{r}\tilde{u}_\theta = \frac{\nu}{k^2}\left(DD' - k^2 - \frac{\sigma}{\nu}\right)(DD'\tilde{u}_r)$$

or

$$\boxed{\frac{\nu}{k^2}\left(DD' - k^2 - \frac{\sigma}{\nu}\right)\left(DD' - k^2\right)\tilde{u}_r - 2\frac{U}{r}\tilde{u}_\theta = 0}$$

NOW WE HAVE TWO EQUATIONS FOR TWO UNKNOWNS $\tilde{u}_r$ AND $\tilde{u}_\theta$ **EQS IN "BOXES"**

$\boxed{\text{CH 12 P2 CONT.}}$

IF $d = R_2 - R_1 << (R_2, R_1)$ THEN $D' \sim D$

ALSO $V(r) = Ar + B/r$ (SEE TEXT FOR A AND B!)

THEN $(D'V) = 2A$

OUR TWO EQUATIONS BECOME:

$$(D^2 - k^2 - \tfrac{\sigma}{\nu})(D^2 - k^2)\tilde{u}_r = \frac{2k^2}{\nu}\frac{V(r)}{r}\tilde{u}_\theta$$

$$(D^2 - k^2 - \tfrac{\sigma}{\nu})\tilde{u}_\theta = \frac{2A}{\nu}\tilde{u}_r$$

$$= -2\frac{(\Omega_1 R_1^2 - \Omega_2 R_2^2)}{(R_2^2 - R_1^2)}\frac{1}{\nu}\tilde{u}_r$$

WHERE $\frac{V(r)}{r} = A + B/r^2$.

• STEP #3: LET'S NORMALIZE VARIABLES AND PARAMETERS

$$\frac{\sigma d^2}{\nu} \to \sigma' \quad k \to k'/d \quad D \to \frac{1}{d}\frac{\partial}{\partial x} = \frac{1}{d}D'$$

AND $xd + R_1 = r$ SO THAT

$$A + \frac{B}{r^2} \cong A + \frac{B}{R_1^2} - \frac{2B}{R_1^3}d\,x$$

$$\boxed{A + \frac{B}{R_1^2} = \Omega_1} \quad \text{AND} \quad -\frac{2B}{R_1^3}d \approx \Omega_1\left(\frac{\Omega_2}{\Omega_1} - 1\right)\cancel{\left(\frac{R_2}{R_1}\right)^{2}}$$

WE OBTAIN:

$$(D^2 - k^2 - \sigma)(D^2 - k^2)\tilde{u}_r = \frac{2k^2}{\nu}d^2\Omega_1(1 + \alpha x)\tilde{u}_\theta$$

$$(D^2 - k^2 - \sigma)\tilde{u}_\theta = -2\frac{(\Omega_1 R_1^2 - \Omega_2 R_2^2)}{R_2^2 - R_1^2}\frac{d^2}{\nu}\tilde{u}_r$$

IF WE <u>NORMALIZE</u> $\frac{2k^2}{\nu}d^2\Omega_1\tilde{u}_\theta \to \tilde{u}_\theta'$ THEN WE OBTAIN:

TA

$$(D^2 - k^2 - \sigma)u_\theta' = -4\frac{(\Omega_1 R_1^2 - \Omega_2 R_2^2)}{R_2^2 - R_1^2}\frac{d^4\Omega_1}{\nu^2}k^2\tilde{u}_r \quad \underline{\underline{\text{QED}}}$$

first of the two integrals included in I_4 is positive definite for $\mu > 0$; but the second is complex. However, the real part of I_4 is positive definite for $\mu > 0$; in fact,

$$\mathrm{re}(I_4) = \int_\eta^1 r\phi(r)\left|\frac{dv}{dr}-\frac{v}{r}\right|^2 dr. \tag{185}$$

For, expanding the integrand in (185), we have

$$\int_\eta^1 r\phi(r)\left|\frac{dv}{dr}-\frac{v}{r}\right|^2 dr = \int_\eta^1 \phi(r)\left(r\left|\frac{dv}{dr}\right|^2+\frac{|v|^2}{r}\right)dr - \int_\eta^1 \phi(r)\frac{d|v|^2}{dr}\,dr; \tag{186}$$

but

$$\int_\eta^1 \phi(r)\frac{d|v|^2}{dr}\,dr = (1-\mu)\int_\eta^1 \left(\frac{1}{r^2}-\kappa\right)\frac{d|v|^2}{dr}\,dr = (1-\mu)\int_\eta^1 \frac{1}{r^2}\frac{d|v|^2}{dr}\,dr. \tag{187}$$

Therefore, the right-hand side of (186) is, indeed, the real part of I_4.

Returning to equation (180) and equating the real parts of this equation, we obtain

$$\mathrm{re}(\sigma)(I_1-\mathscr{T}a^2I_3)+I_2-\mathscr{T}a^2[a^2I_3+\mathrm{re}(I_4)] = 0. \tag{188}$$

When $\mu > \eta^2$, $\mathscr{T} < 0$ and the coefficient of $\mathrm{re}(\sigma)$ in equation (188) is positive definite; and so also are the remaining terms in the equation. Therefore,

$$\mathrm{re}(\sigma) < 0 \quad \text{for } \mu > \eta^2, \tag{189}$$

and the flow is stable; this result is entirely to be expected on physical grounds. Nevertheless, it appears to be the only one which can be established by general analytical arguments. In particular, it does not seem that one can deduce the general validity of the principle of the exchange of stabilities for this problem. For example, by equating the imaginary parts of equation (180), we obtain (cf. equation (176))

$$\mathrm{im}(\sigma)(I_1+\mathscr{T}a^2I_3) = -2Ta^2\,\mathrm{im}\int_\eta^1 \frac{v}{r^2}\frac{dv^*}{dr}\,dr, \tag{190}$$

and no general conclusions can be drawn from this equation; when $\mu < 0$, even I_3 is not positive definite!

71. The solution for the case of a narrow gap when the marginal state is stationary

If the gap R_2-R_1 between the two cylinders is small compared to their mean radius $\frac{1}{2}(R_2+R_1)$, we need not (as in § 68 (b)) distinguish

(FROM S. CHANDRASEKHAR
"HYDRODYNAMIC AND HYDROMAGNETIC
STABILITY"

between D and D_* in equations (161) and (162); and we can also replace $(A+B/r^2)$ which occurs on the right-hand side of equation (161) by

$$\Omega_1\left[1-(1-\mu)\frac{r-R_1}{R_2-R_1}\right]. \tag{191}$$

In rewriting equations (161) and (162) in the framework of these approximations, it will be convenient to measure radial distances from the surface of the inner cylinder in the unit $d = R_2-R_1$. Thus, letting

$$\zeta = (r-R_1)/d, \quad k = a/d, \quad \text{and} \quad \sigma = pd^2/\nu, \tag{192}$$

we have to consider the equations

$$(D^2-a^2-\sigma)(D^2-a^2)u = \frac{2\Omega_1 d^2}{\nu}a^2[1-(1-\mu)\zeta]v \tag{193}$$

and

$$(D^2-a^2-\sigma)v = \frac{2Ad^2}{\nu}u. \tag{194}$$

By the further transformation

$$u \to \frac{2\Omega_1 d^2a^2}{\nu}u, \tag{195}$$

the equations become

$$(D^2-a^2-\sigma)(D^2-a^2)u = (1+\alpha\zeta)v \tag{196}$$

and

$$(D^2-a^2-\sigma)v = -Ta^2u, \tag{197}$$

where, now,

$$T = -\frac{4A\Omega_1}{\nu^2}d^4 \tag{198}$$

and

$$\alpha = -(1-\mu). \tag{199}$$

Equations (196) and (197) must be considered together with the boundary conditions

$$u = Du = v = 0 \quad \text{for } \zeta = 0 \text{ and } 1. \tag{200}$$

We are primarily interested in the solutions of equations (196) and (197) (subject to the boundary conditions (200)) for various values of a for which the real part of σ is zero. The method described in Chapter III, § 31 in a different connexion is applicable to this problem. Thus, we must obtain solutions for two cases: when σ is zero and the marginal state is stationary and when σ is imaginary and the marginal state is oscillatory. In the latter case for each value of a, $i\sigma$ must be determined by the condition that T is real.† In either case, we must find the minimum of T as a function of a; and depending on which of the two minima is lower, we shall have the onset of instability as a stationary secondary flow or as overstability. Careful experiments on the onset of instability

† It is, of course, possible that under certain circumstances solutions with this property do not exist.

(OXFORD PRESS, 1961)

by Taylor and others have failed to reveal any suggestions of overstability. For this reason, the case $\sigma = 0$ is the only one which has been considered in the literature. However, as no general arguments for the validity of the principle of the exchange of stabilities have been found for this problem, the case of overstability requires investigation. We return to this question in § 72.

When the marginal state is stationary, the equations to be solved are

$$(D^2-a^2)^2u = (1+\alpha\zeta)v \tag{201}$$

and

$$(D^2-a^2)v = -Ta^2u, \tag{202}$$

together with the boundary conditions (200).

(*a*) *The solution of the characteristic value problem for the case* $\sigma = 0$

It can be readily verified that the characteristic value problem presented by equations (200)–(202) is not self-adjoint in the usual sense. For this reason, the method to be described below was patterned after the ones which have been found successful in cases where the problems are self-adjoint. However, Roberts has recently found a variational basis for the method; this is considered in Appendix IV.

The method of solution we shall adopt is the following.

Since v is required to vanish at $\zeta = 0$ and 1, we expand it in a sine series of the form

$$v = \sum_{m=1}^{\infty} C_m \sin m\pi\zeta. \tag{203}$$

Having chosen v in this manner, we next *solve* the equation,

$$(D^2-a^2)^2u = (1+\alpha\zeta)\sum_{m=1}^{\infty} C_m \sin m\pi\zeta, \tag{204}$$

obtained by inserting (203) in (201), and arrange that the solution satisfies the four remaining boundary conditions on u. With u determined in this fashion and v given by (203), equation (202) will lead, as we shall presently see, to a secular equation for T.

The solution of equation (204) is straightforward. The general solution can be written in the form

$$u = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2+a^2)^2}\Big\{A_1^{(m)}\cosh a\zeta + B_1^{(m)}\sinh a\zeta + A_2^{(m)}\,\zeta\cosh a\zeta + \\ + B_2^{(m)}\,\zeta\sinh a\zeta + (1+\alpha\zeta)\sin m\pi\zeta + \frac{4\alpha m\pi}{m^2\pi^2+a^2}\cos m\pi\zeta\Big\}, \tag{205}$$

where the constants of integration $A_1^{(m)}$, $A_2^{(m)}$, $B_1^{(m)}$, and $B_2^{(m)}$ are to be

determined by the boundary conditions $u = Du = 0$ at $\zeta = 0$ and 1. These latter conditions lead to the equations:

$$A_1^{(m)} = -\frac{4m\pi\alpha}{m^2\pi^2+a^2}, \qquad aB_1^{(m)} + A_2^{(m)} = -m\pi,$$

$$A_1^{(m)}\cosh a + B_1^{(m)}\sinh a + A_2^{(m)}\cosh a + B_2^{(m)}\sinh a = (-1)^{m+1}\frac{4m\pi\alpha}{m^2\pi^2+a^2},$$

$$A_1^{(m)}a\sinh a + B_1^{(m)}a\cosh a + A_2^{(m)}(\cosh a + a\sinh a) + \\ + B_2^{(m)}(\sinh a + a\cosh a) = (-1)^{m+1}(1+\alpha)m\pi. \tag{206}$$

On solving these equations, we find that

$$A_1^{(m)} = -\frac{4\alpha m\pi}{m^2\pi^2+a^2},$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\{a + \beta_m(\sinh a + a\cosh a) - \gamma_m \sinh a\},$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\{\sinh^2 a + \beta_m a(\sinh a + a\cosh a) - \gamma_m a\sinh a\},$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\{(\sinh a\cosh a - a) + \beta_m a^2\sinh a - \gamma_m(a\cosh a - \sinh a)\}, \tag{207}$$

where

$$\Delta = \sinh^2 a - a^2,$$

$$\beta_m = \frac{4\alpha}{m^2\pi^2+a^2}[(-1)^{m+1} + \cosh a],$$

and

$$\gamma_m = (-1)^{m+1}(1+\alpha) + \frac{4\alpha}{m^2\pi^2+a^2}a\sinh a. \tag{208}$$

Now substituting for v and u from equations (203) and (205) in equation (202), we obtain

$$\sum_{n=1}^{\infty} C_n(n^2\pi^2+a^2)\sin n\pi\zeta \\ = Ta^2\sum_{m=1}^{\infty}\frac{C_m}{(m^2\pi^2+a^2)^2}\Big\{A_1^{(m)}\cosh a\zeta + B_1^{(m)}\sinh a\zeta + A_2^{(m)}\zeta\cosh a\zeta + \\ + B_2^{(m)}\zeta\sinh a\zeta + (1+\alpha\zeta)\sin m\pi\zeta + \frac{4\alpha m\pi}{m^2\pi^2+a^2}\cos m\pi\zeta\Big\}. \tag{209}$$

Multiplying equation (209) by $\sin n\pi\zeta$ and integrating over the range of ζ, we obtain a system of linear homogeneous equations for the constants

$\mathscr{C}_m = C_m/(m^2\pi^2+a^2)^2$; and the requirement that these constants are not all zero leads to the secular equation

$$\Bigg\| \frac{n\pi}{(n^2\pi^2+a^2)}\Big\{[1+(-1)^{n+1}\cosh a]A_1^{(m)}+[(-1)^{n+1}\sinh a]B_1^{(m)}+ +(-1)^{n+1}\Big[\cosh a-\frac{2a}{n^2\pi^2+a^2}\sinh a\Big]A_2^{(m)}+ +\Big[(-1)^{n+1}\sinh a-\frac{2a}{n^2\pi^2+a^2}\{1+(-1)^{n+1}\cosh a\}\Big]B_2^{(m)}\Big\}+ +\alpha X_{nm}+\tfrac{1}{2}\delta_{nm}-\tfrac{1}{2}(n^2\pi^2+a^2)^3\frac{\delta_{nm}}{a^2T}\Bigg\| = 0, \quad (210)$$

$$\text{where } X_{nm} = \begin{cases} 0 \text{ if } m+n \text{ is even and } m \neq n, \\ \tfrac{1}{4} \text{ if } m = n, \\ \dfrac{4nm}{n^2-m^2}\Big\{\dfrac{2}{m^2\pi^2+a^2}-\dfrac{1}{\pi^2(n^2-m^2)}\Big\} \text{ if } m+n \text{ is odd.} \end{cases} \quad (211)$$

On using the first two equations of (206), equation (210) simplifies to the form

$$\Bigg\| \frac{n\pi}{n^2\pi^2+a^2}\Big\{\frac{4m\pi\alpha}{m^2\pi^2+a^2}[(-1)^{m+n}-1]- -\frac{2a}{n^2\pi^2+a^2}[(-1)^{n+1}\{A_2^{(m)}\sinh a+B_2^{(m)}\cosh a\}+B_2^{(m)}]\Big\}+ +\alpha X_{nm}+\tfrac{1}{2}\delta_{nm}-\tfrac{1}{2}(n^2\pi^2+a^2)^3\frac{\delta_{nm}}{a^2T}\Bigg\| = 0; \quad (212)$$

and on substituting for the constants $A_2^{(m)}$ and $B_2^{(m)}$ their explicit solutions given in (207), we find that equation (212) simplifies greatly and we are left with

$$\Bigg\| \frac{4mn\pi^2\alpha}{(n^2\pi^2+a^2)(m^2\pi^2+a^2)}[(-1)^{m+n}-1]- -\frac{2amn\pi^2}{(n^2\pi^2+a^2)^2(\sinh^2 a-a^2)}\Big\{(\sinh a\cosh a-a)[1+(1+\alpha)(-1)^{m+n}]+ +(\sinh a-a\cosh a)[(-1)^{n+1}+(1+\alpha)(-1)^{m+1}]- -\frac{4a\alpha\sinh a}{(m^2\pi^2+a^2)}[\sinh a+a(-1)^{m+1}][(-1)^{m+n}-1]\Big\}+ +\tfrac{1}{2}\delta_{nm}+\alpha X_{nm}-\tfrac{1}{2}(n^2\pi^2+a^2)^3\frac{\delta_{mn}}{a^2T}\Bigg\| = 0. \quad (213)$$

A first approximation to the solution of equation (213) is obtained by setting the (1, 1)-element of the matrix equal to zero. We find

$$\tfrac{1}{2}(\pi^2+a^2)^3\frac{1}{Ta^2} = \tfrac{1}{4}\alpha+\tfrac{1}{2}- -\frac{2a\pi^2(2+\alpha)}{(\pi^2+a^2)^2(\sinh^2 a-a^2)}[(\sinh a\cosh a-a)+(\sinh a-a\cosh a)]. \quad (214)$$

On further simplification, this gives

$$T = \frac{2}{2+\alpha}\,\frac{(\pi^2+a^2)^3}{a^2\{1-16a\pi^2\cosh^2\tfrac{1}{2}a/[(\pi^2+a^2)^2(\sinh a+a)]\}}. \quad (215)$$

We observe that, apart from the factor $2/(2+\alpha)$, this expression for T is identical with what was found in Chapter II (§ 17, equation (311)) for the Rayleigh number for the simple Bénard problem by the variational method in the first approximation for the case of two rigid boundaries. Consequently, in this approximation (cf. equation (199)),

$$T_c = \frac{2}{2+\alpha}\times 1715 = \frac{3430}{1+\mu} \quad \text{and} \quad a_{\min} = 3{\cdot}12. \quad (216)$$

We shall see below (§ (*b*)) that for $0 < \mu < 1$ equation (216) gives values for the critical Taylor number which do not differ from those obtained in the higher approximations by more than one per cent. The reason for this relatively high accuracy of the solution in the first approximation will be made apparent in § (*d*).

(*b*) *Numerical results*

A method of solving the infinite order characteristic equation which (213) provides for T would be to set the determinant formed by the first n rows and columns of the secular matrix equal to zero and let n take increasingly larger values. In practice, the usefulness of this method will depend largely on how rapidly the lowest positive root of the resulting equation of order n tends to its limit as $n \to \infty$. It appears that for the problem on hand, the process converges quite rapidly.

In Table XXXII the values of T obtained with the aid of equation (213) in the different approximations are listed for those values of a (for the assigned μ's) at which it was found (by trial and error) that T attained its minimum value. From an examination of this table, it would appear that for $\mu > -1{\cdot}0$, the third approximation provides T to well within one per cent of the true value. For $-3{\cdot}0 \leqslant \mu \leqslant -1{\cdot}0$, the calculations were carried out to as high approximations as seemed

Ch 12 P3 THIS PROBLEM ASKS TO REPEAT THE ANALYSIS OF CHANDRASEKHAR AND DETERMINE THE CRITICAL WAVELENGTH FOR THE ONSET OF CENTRIFUGAL INSTABILITY.

CHANDRASEKHAR'S ANALYSIS IS ATTACHED.

S. CHANDRASEKHAR WAS A PLASMA ASTROPHYSICIST, AND HE WON THE 1983 NOBEL PRIZE FOR PHYSICS FOR HIS WORK ON STELLAR EVOLUTION.

Ch 12 P5

(i) THE RAYLEIGH EQUATION IS LINEAR IN THE STREAM FUNCTION $\bar{u} = \hat{z} \times \nabla \varphi$. SINCE $u_x = \partial\varphi/\partial y$ AND $u_y = -\partial\varphi/\partial x = -jk\varphi$ THEN u_y AND φ MUST SATISFY THE SAME EQUATION. u_y AND φ ALSO SATISFY THE SAME BOUNDARIES SINCE $\tilde{u}_y$ MUST VANISH AT WALLS.

(ii) IF c IS AN EIGENVALUE, THEN

$$(u_0 - c)(\tilde{u}_y'' - k^2 \tilde{u}_y) - u_0'' \tilde{u}_y = 0$$

WHERE $\tilde{u}_y$ IS A COMPLEX PHASOR. THE COMPLEX CONJUGATE OF THE ABOVE EQ IS:

$$(u_0 - c^*)(\tilde{u}_y^{*\prime\prime} - k^2 \tilde{u}_y^*) - u_0'' \tilde{u}_y^* = 0$$

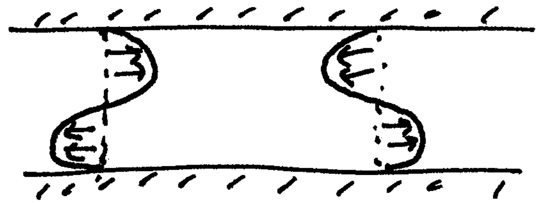
BUT $\tilde{u}_y^*$ SATISFIES THE SAME EQUATION AND BOUNDARY CONDITIONS. THIS MEANS THAT $c^* = c_r - i c_i$ MUST <u>ALSO</u> BE AN EIGENVALUE.

THIS PROPERTY OCCURS BECAUSE THE EQUATION (RAYLEIGH'S) DOES NOT DEPEND UPON $i = \sqrt{-1}$.

Ch 12 P 5 CONT

ANTI-SYMMETRIC JET

(iii) FOR AN ANTI-SYMMETRIC JET, WE HAVE



$$U(y) = -U(-y)$$

$$\frac{\partial U}{\partial y} \to -\frac{\partial U}{\partial y}$$

$$\frac{\partial^2 U}{\partial y^2} \to -\frac{\partial^2 U}{\partial y^2}$$

THEN, TRANSFORMING THE EQUATION

TO $y \to -y$, WE HAVE

$$(-U_0 - c)(\tilde{u}_y'' - k^2 \tilde{u}_y) + U_0'' \tilde{u}_y = 0$$

OR $$(U_0 + c)(\tilde{u}_y'' - k^2 \tilde{u}_y) - U_0'' \tilde{u}_y = 0$$

THUS c AND $-c$ MUST BOTH BE EIGENVALUES.

THEREFORE, SOLUTIONS MUST EXIST THAT PROPAGATE IN BOTH DIRECTIONS.

(iv) HERE, WE FOLLOW THE "HINT". FOR A SYMMETRIC JET, $\tilde{u}_y(-y)$ AND $\tilde{u}_y(y)$ SATISFY THE SAME EQUATION. THUS, WE CAN DEFINE:

SYMMETRIC SOLUTION: $S(y) \equiv \tilde{u}_y(y) + \tilde{u}_y(-y)$

ANTI-SYMMETRIC SOLUTION: $A(y) \equiv \tilde{u}_y(y) - \tilde{u}_y(-y)$

COMBINING

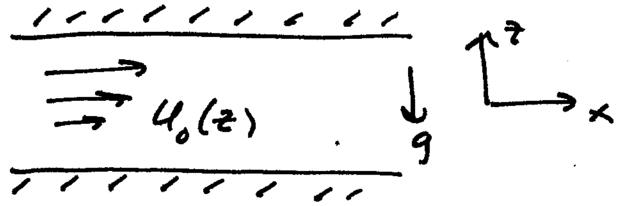
$$A(y) \times [S - EQN] - S(y) \times [A - EQN] = 0$$

RESULTS IN

$$(U_0 - c)(AS'' - SA'') = 0$$

OR $\frac{d}{dy}(AS' - SA') = 0$. THIS GIVES $S \propto A$ WHICH IS IMPOSSIBLE. ONLY ONE OR OTHER.

Ch 17 P.6



THIS QUESTION ASKS US TO REPRODUCE EQ. (7.83) EXCEPT ADDING THE BUOYANCY TERM FROM GRAVITY AND DENSITY STRATIFICATION.

USE NAVIER-STOKES (EULER) AND MASS CONSERVATION:

$$\#1: \quad \frac{\partial \tilde{u}}{\partial t} + u_0 \frac{\partial \tilde{u}}{\partial x} + \hat{x}\, \tilde{u}_z \frac{\partial u_0}{\partial z} + \frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0} g \hat{z} = 0$$

$$\#2: \quad \frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + u_0 \frac{\partial \tilde{\rho}}{\partial z} + \tilde{u}_z \frac{\partial \rho_0}{\partial z} = 0$$

$$\text{BUT} \quad \omega_B^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} = \text{BUOYANCY FREQUENCY}^2$$

- COMBINING:

$$\tilde{u}\cdot(\#1) \Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}\tilde{u}^2\right) + u_0 \frac{\partial}{\partial x}\left(\tfrac{1}{2}\tilde{u}^2\right) + \tilde{u}_x \tilde{u}_z \frac{\partial u_0}{\partial z} + \frac{1}{\rho_0}\tilde{u}\cdot\nabla p + \frac{\rho}{\rho_0} g \tilde{u}_z = 0$$

$$\frac{g^2 \rho}{\rho_0^2 \omega_B^2} \times (\#2) \Rightarrow \frac{g^2}{\rho_0^2 \omega_B^2}\left[\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho^2\right) + \rho\, u_0 \frac{\partial \tilde{\rho}}{\partial x}\right] - \tilde{u}_z \frac{g\rho}{\rho_0} = 0$$

- ADD

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\frac{g^2\rho^2}{\rho_0^2\omega_B^2}\right) + u_0 \frac{\partial}{\partial x}\left(\tfrac{1}{2}\tilde{u}^2 + \tfrac{1}{2}\frac{g^2\rho^2}{\rho_0^2\omega_B^2}\right) + \underbrace{\frac{1}{\rho_0}\tilde{u}\cdot\nabla p}_{\frac{1}{\rho_0}\left[\nabla\cdot(\tilde{u}p) - p\nabla\cdot\tilde{u}\right]} + \tilde{u}_x \tilde{u}_z \frac{\partial u_0}{\partial z} = 0$$

- FINALLY INTEGRATE OVER CONTROL VOLUME AS ON P. 502 USE BOUNDARY CONDITIONS TO ELIMINATE MOST TERMS AND THIS RESULTS IN GLOBALLY-INTEGRATED ENERGY EQUATION!

Ch 13 P1

$$S(\omega) = \frac{1}{2\pi}\int e^{-j\omega t} R(t)\,dt$$

$$= \frac{1}{2\pi}\int (\cos\omega t - i\sin\omega t)\, R(t)\,dt$$

IF $R(t) = R(-t)$, THEN THE TERM WITH $\sin\omega t$ VANISHES.

IF $R(t)$ IS ALSO REAL, THEN $S(\omega)$ IS REAL AND SYMMETRIC.

Ch 13 P2

$$u(t) = u_0 \cos t + \bar{u} \qquad (t \to \omega t)$$

$$\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} dt\, u(t) = \bar{u}$$

$$\overline{u^2} = \frac{1}{2\pi}\int_0^{2\pi} dt\,\left(u_0^2\cos^2 t + 2u_0\bar{u}\cos t + \bar{u}^2\right)$$

$$= \frac{1}{2}u_0^2 + \bar{u}^2$$

$$u_{RMS} = \sqrt{\overline{u^2}} = \sqrt{\tfrac{1}{2}u_0^2 + \bar{u}^2}$$

$$u_{STD}^2 = \overline{(u-\bar{u})^2} = \frac{1}{2\pi}\int_0^{2\pi} dt\; u_0^2\cos^2 t$$

$$= \frac{1}{2}u_0^2$$

SO $u_{STD} = u_0/\sqrt{2}$

Ch 13 P3

$$R(\tau) = \overline{u(t)\,u(t+\tau)}$$

$$= \frac{u^2}{2\pi}\int_0^{2\pi} \cos t' \cos(t+\tau)\,dt'$$

BUT $\cos(t)\cos(t+\tau) = \cos(t)\left[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)\right]$

AND $R(t) = \dfrac{u^2}{2}\cos(\tau)$ ALSO PERIODIC

Ch 13 P5

HEAT FLUX $= \rho C_p \overline{\tilde{w}\tilde{T}} = \rho C_p \frac{1}{2}\tilde{w}_{RMS}\tilde{T}_{RMS}$

AIR 20°C $\Rightarrow$ $C_p \sim 1012\ J/kg\,°K$ $\rho = 1.2\,kg/m^3$ SO HEAT FLUX $= 61\ w/m^2$

Ch 13 P10

UNDERSTANDING TRANSPORT CAUSED BY TURBULENCE IS A CHALLENGE. WHEN THE FLUCTUATIONS CAUSE RANDOM MOTION, THE STATISTICS ARE GAUSSIAN AND RELATIVELY EASY TO DESCRIBE.–

$$\frac{d}{dt}\overline{x^2} = 2\,\overline{x\frac{dx}{dt}} \qquad \text{BUT } x = \int_0^t dt'\,u(t')$$

$$= 2\,\overline{u^2}\int_0^t R(\tau)\,dt$$

AT LONG TIMES, THE AVERAGE OF X FROM IT'S ORIGIN AT $t=0$ SCALES LIKE

$$(\Delta x)^2 \sim \left(\overline{u}^2\, 2\, \mathcal{T}_c\right) t$$

WHERE $\mathcal{T}_c \equiv \int_0^\infty d\tau\, e^{-t^2/\tau_c^2} = \frac{\sqrt{\pi}}{2}\tau_c$

CORRELATION TIME

THE PARTICLE DIFFUSIVITY SCALES LIKE:

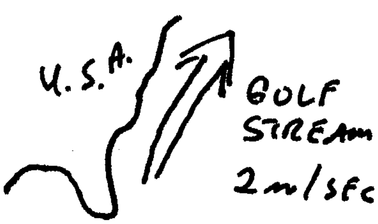
$$K_p \sim \frac{1}{2}\frac{d}{dt}(\Delta x)^2 \sim u_{rms}^2\,\mathcal{T}_c$$

WITH $t_c = 1s$ $\quad u_{rms} = 1m/s$ $\quad K_p \sim 0.89\ m^2/s$

Ch 14 P1

GEOSTROPHIC BALANCE

$f \sim 2\omega \sin(45°N) \sim 10^{-4}\ \text{sec}^{-1}$



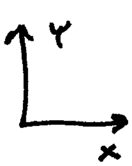
$$-f u_y = \frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

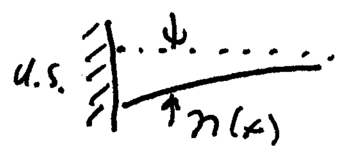
$g = 9.8\ m/s^2$

WITH $P = \rho g(H + \eta - z)$

$$\frac{\partial P}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{f u_y}{g} = 2\times10^{-5}$$

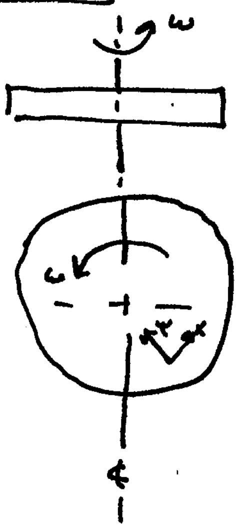
$$\approx 2\ m/km$$





CORIOLIS FORCE DEPRESSES WATER LEVEL.

Ch 14 P2



δ_E = EKMAN LAYER

$$= \sqrt{\frac{2\nu}{f}}$$

WITHIN VISCOUS EKMAN LAYER, CORIOLIS FORCE IS BALANCED BY VISCOSITY

$$\hat{x}: \quad -f u_y = \nu \frac{\partial^2 u_x}{\partial z^2}$$

$$\hat{y}: \quad f u_x = \nu \frac{\partial^2 u_y}{\partial z^2}$$

THESE EQS ARE SOLVED IN SEC. 6 IN TEXT

SCALE OF LENGTH MUST BE $\propto \sqrt{\nu/f}$

FOR THIS PROBLEM, $\nu = 10^{-6}$ $\quad \omega = 2\pi/6$ $\quad f = 2\omega$

SO $\quad \delta_E \sim \sqrt{\frac{2\ 10^{-6}\ 6}{2\pi}} = 3 \times 10^{-3}\ m$ (3 mm) VERY THIN

Ch 14 P3

WITHIN AN EKMAN LAYER IS A POLAR FLOW $\int_0^\delta u_y\, dz \sim \frac{1}{2} u \delta$

HERE, $u \sim 10\ m/s$, AND $\delta = \sqrt{2\nu/f} = \sqrt{\frac{2 \cdot 10}{10^{-4}}} \sim 440\,m$

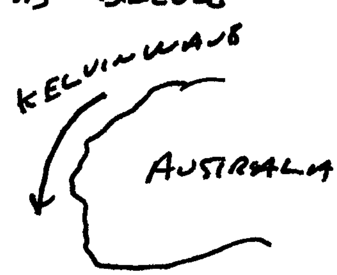
SO $\frac{1}{2} u\delta \sim 2 \times 10^3\ m^2/sec$, A SIGNIFICANT FLUX.

Ch 14 P5

AS EXPLAINED ON P. 616 IN TEXT, KELVIN WAVES PROPAGATE WITH COAST ON THE LEFT IN SOUTHERN HEMISPHERE. THUS, WAVES PROPAGATE AS BELOW

WITH $H \sim 50\ m$ FOR THERMOCLINE

$\rho \sim 1000\ kg/m^3$



$$C = \sqrt{\underbrace{g\frac{\Delta\rho}{\rho}}_{\text{REDUCED } g \text{ IN CONTINUOUSLY STRATIFIED}} H} = 1\ m/sec$$

WITH $f \sim 2\omega \sin 30°$

$$\Lambda = C/f \sim 14\ km$$

Ch14 P7

(i)

$H \sim 10\,\text{km}$ $\quad \Delta T \sim 50^\circ C \quad \theta \sim 45^\circ N$

$$\omega_B^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \qquad \text{BUT} \quad \rho = \rho_0\left(1 - \alpha(T - T_0)\right)$$

$$\frac{\partial \rho}{\partial z} \sim -\alpha \rho_0 \frac{\partial T}{\partial z} \propto \frac{\Delta T}{H}$$

$$= g\alpha \frac{\Delta T}{H}$$

$$\simeq 10^{-4}$$

$$\text{So} \quad \omega_B \sim 10^{-2}\ \text{RAD/SEC} \qquad \frac{2\pi}{\omega} = 518\ \text{SEC}\ (8\ \text{min})$$

(ii) FROM P.608 IN TEXT

$$c \sim \frac{\omega_B H}{\pi} \sim 41\ \text{m/s}$$

(iii) $c_{Rossby} \sim -\beta \dfrac{c^2}{s^2}$ $\quad$ (FOR $k \sim \frac{1}{R}$)

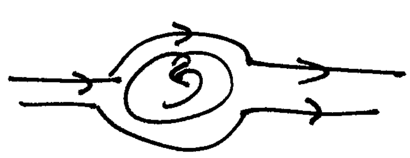
$$\beta \sim \frac{2\pi\,\omega\cos\theta}{R} \sim 2\times10^{-11} \qquad \text{SO} \quad c_{Rossby} \sim 3.4\ \text{m/SEC}$$

Ch14 P.8

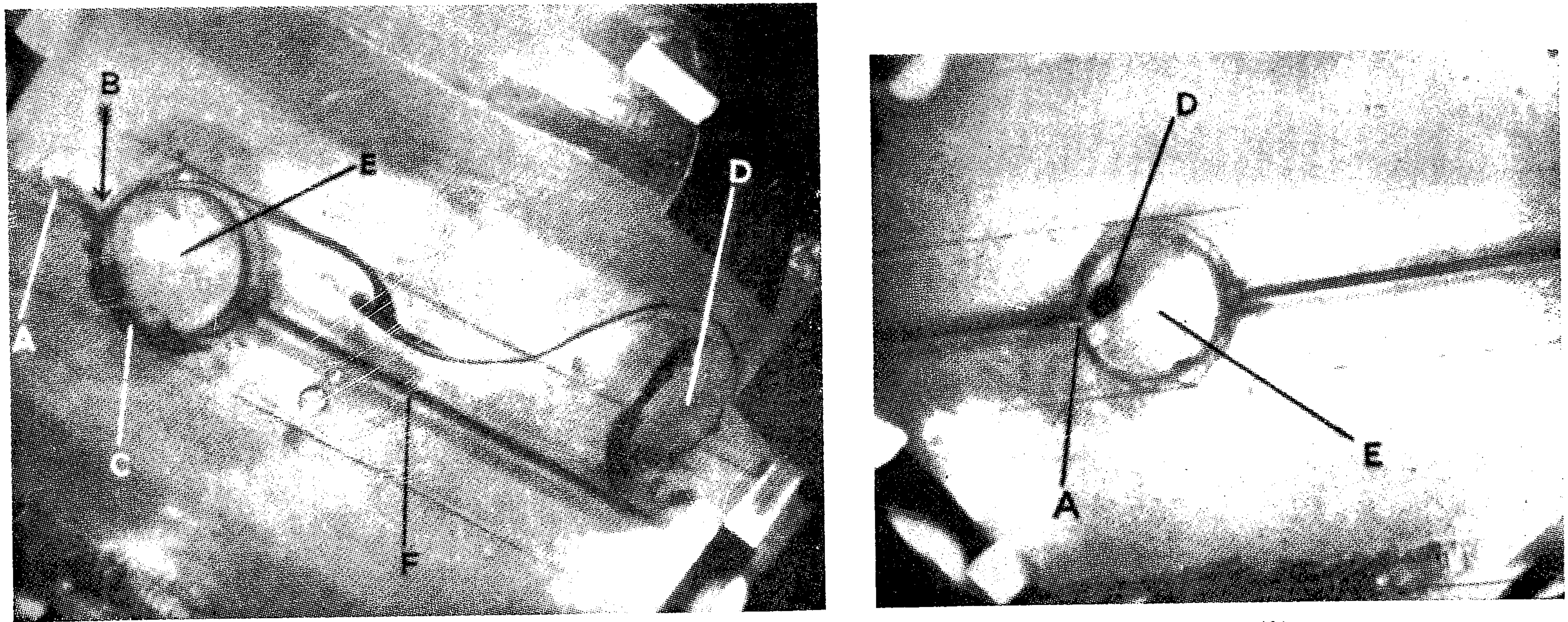
THIS BEAUTIFUL PROBLEM RELATES TO AN EXPERIMENT CONDUCTED BY G.I. TAYLOR IN 1923. ALSO TO TAYLOR-PROUDMAN COLUMNS DESCRIBED IN SEC. 4 OF Ch. 14.

WHEN THE HYDROSTATICS ARE DOMINATED BY THE CORIOLIS FORCE, THEN THE HORIZONTAL FLOW IS <u>INDEPENDENT</u> OF z.

THIS IS EQ. 14-21 CALLED THE <u>TAYLOR-PROUDMAN</u> <u>THEOREM</u>! IN THIS CASE, THE FLOW, u, IS DIVERTED <u>AROUND</u> THE ROTATING CYLINDER AS IF IT ACTUALLY EXTENDS <u>UPWARD</u>!



SEE DISCUSSION AND PHOTOS OF AN ACTUAL EXPERIMENT ATTACHED!



(a) (b)

Figure 7.6.3. Motion in a rotating dish of water 4 in. deep due to slow translation of a circular cylinder (E) of height 1 in. from right to left across the bottom of the dish, viewed from above. In (a) the dye has been released at a (moving) point A above the top of the cylinder and directly ahead of it, and B (a dividing point), C and D are subsequent positions of the dye; in (b) the point of release (A) is within the upward projection of the cylinder and the dye remains in the blob D. The flow evidently has a two-dimensional character. (From Taylor 1923.)

FROM "AN INTRODUCTION TO FLUID DYNAMICS"
BY G. K. BATCHELOR CAMBRIDGE PRESS 1967

is to say, if the basic rotation in the lateral plane is anti-clockwise, the Coriolis force tends to turn the direction of motion of an element relative to the rotating frame to the right. Moreover, the Coriolis force is linear in the velocity, and will tend to change the direction of the component of **u** in a lateral plane at the same rate for all magnitudes and directions of that component. Thus a material element whose motion is dominated by the Coriolis force moves on a path whose projection on a lateral plane is a circle, the whole circle being completed in a time of order Ω^{-1}. The Coriolis force evidently tends to restore an element to its initial position in the lateral plane. Note that there is no special significance about the position of the axis of rotation, so far as the Coriolis force is concerned.

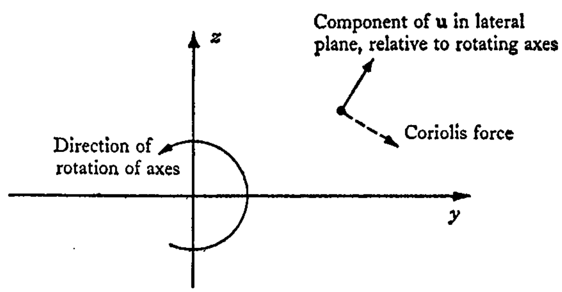


Figure 7.6.1. To show the direction of the Coriolis force which acts in a rotating reference system. The (y, z)-plane is normal to the axis of rotation.

Since the motions of different elements of the fluid normally exercise a strong mutual influence through the action of pressure gradients, it is desirable to consider also the collective effect of Coriolis forces on different elements. Suppose that relative to the rotating axes there is generated a motion which leads to a non-zero and positive value of the rate of expansion in the lateral plane, that is, to a positive value of

$$\partial v/\partial y + \partial w/\partial z$$

(in terms of the co-ordinate system of figure 7.6.1) over a certain region of the fluid. The area enclosed by the projection on a lateral plane of a closed material curve in this region will then be increasing. The effect of the Coriolis force accompanying this general outward movement of the projected curve is to generate a tangential motion of the material curve, which makes a *negative* contribution to the circulation round it. This change in the circulation of the motion relative to rotating axes is of course simply the change required to keep the circulation relative to an absolute frame constant during a movement leading to increase of the projected area on a lateral plane. Now the new tangential motion of the material curve itself leads to a Coriolis force in a direction normal to the curve; and inasmuch as the new tangential motion makes a net negative contribution to the circulation, the associated Coriolis force is directed mostly inwards and so tends to produce a *reduction*

in the area enclosed by the projected curve. In other words, at places in the fluid where there is a positive value of $\partial v/\partial y + \partial w/\partial z$ the effect of Coriolis forces is to tend to produce a negative value, and vice versa. Thus the net effect of Coriolis forces is to oppose displacements of fluid elements which together lead to change of the area enclosed by the projection of a material curve on a lateral plane, that is, to a non-zero expansion in a lateral plane.

The extent to which the restoring effect of Coriolis forces restricts the displacement of fluid elements evidently depends on the relative magnitudes of Coriolis forces and other forces acting on the fluid; and in the present context these other forces are inertia forces. If U is a representative velocity magnitude (relative to rotating axes) and L is a measure of the distance over which **u** varies appreciably, the ratio of the magnitudes of the terms $\mathbf{u}.\nabla\mathbf{u}$ and $2\boldsymbol{\Omega}\times\mathbf{u}$ in (7.6.1) is of order

$$U/L\Omega.$$

The value of this ratio, known as the *Rossby number* in recognition of the work of the Swedish meteorologist, provides a convenient measure of the importance of Coriolis forces. When $U/L\Omega \gg 1$, Coriolis forces are likely to cause only a slight modification of the flow pattern; but when $U/L\Omega \ll 1$, the tendency for Coriolis forces to oppose any expansion in a lateral plane is likely to be dominant. And in the intermediate case when $U/L\Omega$ is of order unity, an interesting mixture of effects is to be expected, some hint of which was provided by the discussion of steady axisymmetric flow with swirl in §7.5.

Steady flow at small Rossby number !!

The dominance of Coriolis forces in flow at values of $U/L\Omega$ small compared with unity has strange consequences when the flow is also steady relative to rotating axes, as was first pointed out by J. Proudman (1916). In steady flow a material element of fluid moves along the same streamline, without reversal of direction, at all times. But the strong Coriolis forces oppose any displacement of fluid elements leading to a non-zero expansion in a lateral plane. It follows that, in the limit $U/L\Omega \to 0$, the form of the streamlines must be consistent with a zero rate of expansion in a lateral plane.

We can establish this result formally by noting that when $\partial\mathbf{u}/\partial t = 0$ and the term $\mathbf{u}.\nabla\mathbf{u}$ is negligible by comparison with the Coriolis force, the equation of motion (7.6.1) becomes

$$2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p, \tag{7.6.2}$$

that is,

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) = (0, 2\Omega w, -2\Omega v)$$

with the co-ordinates shown in figure 7.6.1. Elimination of p then gives

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.6.3}$$

in steady flow relative to rotating axes, and, as a consequence of the mass-

LIKE EQ. 14-21 IN TEXT

conservation equation,

$$\partial u/\partial x = 0. \quad !! \tag{7.6.4}$$

The curious property of these approximate equations which hold when $U/L\Omega \ll 1$ is that the motion in the lateral or (y, z)-plane is not coupled with the motion parallel to the axis of rotation. Furthermore, none of the flow properties depends on x. Proudman's theorem is sometimes stated as being that 'slow' steady motions relative to rotating axes must be two-dimensional. Since in this book we have regarded the term two-dimensional motion as implying that the velocity vector everywhere lies in a certain plane, it would be more appropriate here to say that steady motions at small Rossby number must be a superposition of a two-dimensional motion in the lateral plane and an axial motion which is independent of x.

The value of the velocity component u parallel to the axis of rotation is evidently determined by the boundary conditions. It will often happen that every line in the fluid parallel to the axis meets a stationary boundary; in such cases the above relations require $u = 0$ everywhere, and only the two-dimensional motion remains. The photographs on plate 23 (made by G. I. Taylor many years before the subject of rotating fluids had attracted much notice) of the flow in an open flat dish of water which is rotating show that Coriolis forces do indeed make the motion two-dimensional in these circumstances. In figure 7.6.2 (plate 23) a drop of coloured liquid has been drawn out into a thin sheet by a 'slow' motion imparted to the rotating fluid, and the two photographs, taken by a camera placed on the axis of rotation of the dish, show that the sheet is everywhere parallel to the axis and that the component of velocity in a lateral plane is independent of x. The flow revealed by the streak of dye released from point A in figure 7.6.3 (plate 23) is more startling. The motion relative to rotating axes is due here to a portion of a circular cylinder E being drawn slowly across the bottom of the dish. The depth of water is 4 in. and the cylinder is 1 in. high, and in a non-rotating fluid the water would pass over the top of the moving cylinder as well as round the sides. However, the dye emerging from a point $1\frac{1}{2}$ in. above the top of the cylinder and directly ahead of it (figure 7.6.3*a*) divides at point B, as if it had met an upward extension of the cylinder, and passes round this imaginary cylinder in two sheets,† the sheet on one side (D) even showing separation and the formation of eddies. In figure 7.6.3*b* the dye is being released from a point just inside the cylindrical region vertically above the body, and collects in a blob which moves with the cylinder. It seems that the flow outside the upward projection of the cylinder is approximately the same as if the cylinder extended from the bottom to the top of the layer of water, and that vertically above the cylinder there is a cylindrical column of water which moves with it. Thus the motion is two-dimensional in the way that is consistent with translation of the cylinder, even though the height of that cylinder is only one-quarter of the depth of water.

READ THIS

† Another striking photograph of this phenomenon is reproduced in Greenspan's book.

When the fluid is not enclosed by stationary boundaries intersected by lines parallel to the axis of rotation, the value of the axial velocity component in the fluid will usually be determined by conditions at an inner boundary. An interesting and fundamental case is flow due to translation of a rigid body, with velocity U parallel to the axis of rotation, through fluid which is unbounded in that direction. It seems that here the above requirements for flow with $U/L\Omega \to 0$ can be satisfied only if all the fluid in the cylinder circumscribing the body moves parallel to the axis with the body, the component of velocity in a lateral plane being zero everywhere. Experiments do in fact suggest that a column of fluid is pushed ahead of a body moving parallel to the axis, although the flow behind the body seems not to be wholly in accord with the above simple theory. Further reference to these experiments will be made later in this section.

In the case of bodies moving either parallel to the axis of rotation or normal to it, the above theory for flow at small Rossby number leads to the conclusion that a so-called 'Taylor column' of fluid parallel to the axis accompanies the body. At the edge of the column there are shear layers where the vorticity is large. It is to be expected that the approximate linear equation (7.6.2) is not applicable everywhere in these layers, although the consequences for the whole flow field are not well understood.

Propagation of waves in a rotating fluid

We have seen that any displacement of the elements of a fluid in rigid-body rotation which leads to a non-zero expansion in a lateral plane is accompanied by Coriolis forces which tend to eliminate this expansion. Since there is no dissipation of energy in an inviscid fluid, it follows that a displacement of this kind which is given to the fluid initially may set up an oscillation. This raises the possibility that a train of waves can propagate through a rotating fluid, with different phases of the wave being associated with positive and negative values of the expansion in a lateral plane. We can examine this possibility by seeking solutions of the equation governing departures from a state of rigid-body rotation which are periodic in time and in certain spatial co-ordinates.

We shall consider first the physically simple case of an axisymmetric wave motion with propagation in the direction of the axis of rotation. Relative to rotating axes, the wave motion is superimposed on stationary fluid and so, for a simple harmonic wave, all flow quantities vary sinusoidally in time with angular frequency β say (period $2\pi/\beta$) and sinusoidally with respect to the axial co-ordinate x with wave-number α say (wavelength $2\pi/\alpha$). The governing equation for flow relative to rotating axes is (7.6.1), in axisymmetric form, and, following the usual pattern of investigation of wave motions, we might proceed by neglecting terms in this equation of degree higher than the first in quantities representing the departure from the undisturbed state. However, there is no need to go through the details, because use can