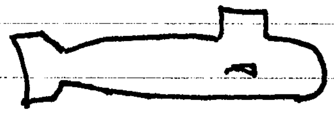
APPH 4200 HW5 : SOLUTIONS

#1 PROBLEM 8.2



1/25TH SCALE
MODEL

WIND TUNNEL TEST: $P = 200\,kPA$ (~ 2 ATM)
$T = 300°K$ ($\sim 27°C$)

WHAT VELOCITY TO BE SIMILAR TO 30 KM/HR AT FULL SCALE?

SINCE MODEL / SUB IS FAR FROM SURFACE, THEY ARE SIMILAR WHEN HAVING EQUAL REYNOLDS NUMBERS. THIS MEANS THE DRAG (VISCOUS) COEFFICIENTS WILL BE EQUAL.

$$Re = \left.\frac{UL}{\nu}\right|_{MODEL} = \left.\frac{UL}{\nu}\right|_{FULL}$$

$$U_{MODEL} = U_{FULL}\left(\frac{L_{FULL}}{L_{MODEL}}\right)\left(\frac{\nu_{MODEL}}{\nu_{FULL}}\right)$$

U_{FULL}: 30 KM/HR; $\frac{L_{FULL}}{L_{MODEL}}$: 25

$$\frac{\nu_{AIR}}{\nu_{WATER}} = \frac{1.6\times10^{-5}}{1.0\times10^{-6}} \approx 16$$

SO $U_{MODEL} = U_{FULL} \times 400$

BUT SPEED OF SOUND IS 1.5 KM/S OR 5,400 KM/HR. THUS, MODEL SPEED MUST BE FASTER THAN SPEED OF SOUND! THIS MAKES THE MODEL TEST INVALID SINCE THE MACH NUMBER WILL NOT BE THE SAME IN BOTH CASES

$$\text{COEFFICIENT OF DRAG} = \left.\frac{D}{\rho U^2 L^2}\right|_{MODEL} = \left.\frac{D}{\rho U^2 L^2}\right|_{FULL}$$

#1 CONTINUED

THUS
$$\frac{D_{MODEL}}{D_{FULL}} = \left(\frac{\rho_{MODEL}}{\rho_{FULL}}\right)\left(\frac{U_m}{U_{FULL}}\right)^2\left(\frac{L_{MODEL}}{L_{FULL}}\right)^2$$

$$= 0.522$$

WHERE $\rho_{MODEL} = (P/RT)_{MODEL} = 2.32\ kg/m^3$

$\rho_{WATER} = 1000\ kg/m^3$

#2 PROBLEM: NON-DIMENSIONALIZE HEAT CONDUCTION EQ

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (u\cdot\nabla)T = \kappa\nabla^2 T$$

NOTE: SINCE THIS EQUATION IS LINEAR IN T, T IS MADE NON-DIMENSIONAL TRIVIALLY.

LET $\nabla^2 \rightarrow \frac{1}{L^2}\nabla'^2 \qquad u \rightarrow U_0\, u'$

$\text{time} \rightarrow L/U_0 \qquad \frac{\partial}{\partial t} \rightarrow \frac{U_0}{L}\frac{\partial}{\partial t'}$

THEN, $\frac{U_0}{L}\left(\frac{\partial T}{\partial t'} + u'\cdot\nabla' T\right) = \frac{\kappa}{L^2}\nabla'^2 T$

OR
$$\frac{\partial T}{\partial t} + u\cdot\nabla T = \underbrace{\frac{\kappa}{U_0 L}}_{P_E = \text{PECLET NUM}}\nabla^2 T$$

$P_E = \frac{UL}{\kappa}$ = RATIO OF $\frac{\text{THERMAL ADVECTION}}{\text{THERMAL DIFFUSION}}$

#3: PROBLEM: HOW DOES VISCOSITY DAMP DEEP WATER GRAVITY WAVES?

REVIEW: THE SURFACE IS MOVING VERTICALLY AS

$$\eta(x,t) = \eta_0 \cos(kx-\omega t)$$

IF WE ASSUME $\nabla \times \bar{u} = 0$ THEN $\bar{u} = \nabla\varphi$ (VELOCITY POTENTIAL). SINCE $\nabla \cdot \bar{u} = 0$, THEN $\nabla^2 \varphi = 0$.

AT THE SURFACE, WE REQUIRE $\frac{\partial \eta}{\partial t} \cong \frac{\partial \varphi}{\partial z}$ AT $z = \eta \approx 0$

AND ALSO BERNOULLI: $\frac{\partial \varphi}{\partial t} + g\eta \approx 0$

THE SOLUTION IS $\omega^2 = kg$ WITH $\varphi(x,z,t) = \varphi_0 e^{kz} \sin(kx-\omega t)$ WITH $\varphi_0 = \eta_0\left(\frac{\omega}{k}\right)$ FOR $z<0$

NOTICE AT THE SURFACE $U_x \sim \frac{\partial \varphi}{\partial x} \sim \varphi_0 k \cos(kx-\omega t)$

AT ANY POINT AT THE SURFACE, THE FLOW OSCILLATES BACK-N-FORTH TANGENTIALLY. THIS WILL GENERATE VISCOUS PERTURBATIONS NEAR THE SURFACE. BERNOULLI'S PRINCIPLE CAN NOT BE APPLIED IN THIS THIN BOUNDARY. (SEE CLASS NOTES FOR NOV 20, 2007.)

HOWEVER, ~~FAR~~ AWAY FROM THE SURFACE, THE FLOW WILL BE IRROTATIONAL, AND $\bar{u} = \nabla\varphi$. SUBSTITUTING THIS INTO NAVIER-STOKES....

$$\frac{\partial \bar{u}}{\partial t} = -g\hat{z} + \nu \nabla^2 \bar{u} \Rightarrow \nabla\left(\frac{\partial \varphi}{\partial t} + gz - \nu\nabla^2\varphi\right) = 0$$

BUT $\nabla^2\varphi = 0$, SO VISCOSITY HAS NO INFLUENCE ON THE FLOW WHEN $\nabla \times \bar{u} = 0$.

#3 CONTINUED

HOW TO ESTIMATE VISCOUS DAMPING FOR WEAK VISCOSITY?

WAVE ENERGY = (MECHANICAL ENERGY OF OSCILLATIONS) + (POTENTIAL ENERGY OF OSCILLATIONS)

$$\text{RATE OF VISCOUS DISSIPATION} = -2\mu \sum_{ij} \epsilon_{ij}^2$$

$$\epsilon_{ij} \rightarrow \epsilon_{xz} = \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$= \frac{\partial^2 \varphi}{\partial x \partial z}$$

$$= -\varphi_0 k^2 \sin(kx - \omega t)\, e^{kz}$$

WE KNOW THAT POTENTIAL AND KINETIC ENERGIES ARE EQUAL, SO

$$\text{WAVE ENERGY} = 2\left(\tfrac{1}{2}\rho u^2\right)$$

$$= \rho\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right]$$

$$= \rho k^2 \varphi_0^2 e^{2kz}$$

$$\text{WAVE (AMPLITUDE) DAMPING RATE} = \gamma = -\frac{1}{2}\frac{(\text{VISCOUS DISSIPATION})}{(\text{WAVE ENERGY})}$$

WHERE WE MUST TIME AVERAGE THE VISCOUS DISSIPATION.

THEREFORE:

$$\gamma = \frac{1}{2}\frac{2\mu\, 2\varphi_0^2 k^4 \overline{\sin^2}}{\rho k^2 \varphi_0^2} = 2k^2\nu$$

$$\text{NOTE: } \gamma/\omega \sim k^2 \underbrace{\left(\frac{2\nu}{\sqrt{kg}}\right)}_{\text{VISCOUS BOUNDARY LAYER} \ll 1}$$

PROBLEM #2 (AGAIN)

$$\bar{u} = \nabla \varphi \qquad \varphi(x, z, t) = \mathrm{Re}\left\{ \varphi \, e^{-j(\omega t - kx)} \, e^{kz} \right\}$$

For $z < 0$

$$\text{WAVE ENERGY} = \iiint dV \left(\tfrac{1}{2} \rho u^2 + \rho g (z - \bar{z}) \right)$$

$$= 2 \times \iiint dV \tfrac{1}{2} \rho u^2$$

$$u^2 \propto \bar{u}^* \cdot \bar{u} = \frac{\partial \varphi^*}{\partial x} \cdot \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi^*}{\partial z} \cdot \frac{\partial \varphi}{\partial z}$$

$$= |\varphi|^2 k^2 e^{2kz} + |\varphi|^2 k^2 e^{2kz}$$

$$\text{DISSIPATION} = \iiint dV \, 2\mu \epsilon_{ij}^2$$

$$= \iiint dV \, 2\mu \, \epsilon_{ij}^* \epsilon_{ij}$$

$$\epsilon_{ij} = \left\{ \epsilon_{xx}, \epsilon_{zz}, \epsilon_{xz}, \epsilon_{zx} = \epsilon_{xz} \right\}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial^2 \varphi}{\partial x^2} = -k^2 \varphi$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{\partial^2 \varphi}{\partial z^2} = -k^2 \varphi$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = \frac{\partial^2 \varphi}{\partial x \partial z} = i k^2 \varphi$$

$$\epsilon_{ij}^* \epsilon_{ij} = k^4 |\varphi|^2 + k^4 |\varphi|^2 + 2k^4 |\varphi|^2 = 4k^4 |\varphi|^2$$

$$\frac{d}{dt}(\text{WAVE ENERGY}) = -2\gamma(\text{WAVE ENERGY})$$

$$= -(\text{DISSIPATION})$$

$$\therefore \quad \gamma \equiv \frac{\text{DISSIPATION}}{2(\text{WAVE ENERGY})}$$

$$\propto \frac{8\nu\rho k^4 |\varphi|^2}{4\rho k^2 |\varphi|^2}$$

$$\gamma = 2\nu k^2$$

THERE FORE

$$\varphi(x, z, t) = \mathcal{R}\left\{ \varphi e^{-i(\omega t - kx)} e^{kz} e^{-\gamma t} \right\}$$

NOTE:

$$\frac{\gamma}{\omega} \sim k^2 \underbrace{\left(\frac{2\nu}{\sqrt{kg}}\right)}_{\delta^2 = \text{BOUNDARY LAYER}}$$

HOW BIG?

$$k = \frac{2\pi}{\lambda} \qquad \lambda = 10\,cm = 0.1\,m \qquad \nu = 10^{-6}\,\frac{m^2}{sec}$$

$$\omega = \frac{2\pi}{T} \qquad T \sim \tfrac{1}{4}\,sec \qquad \omega = \sqrt{kg}$$

$$\frac{2\nu}{\omega} \sim \frac{2\,10^{-6}}{8\pi} \sim 10^{-7}\,m^2 \qquad k^2\left(\frac{2\nu}{\sqrt{kg}}\right) \sim 10^{-3} \ll 1$$