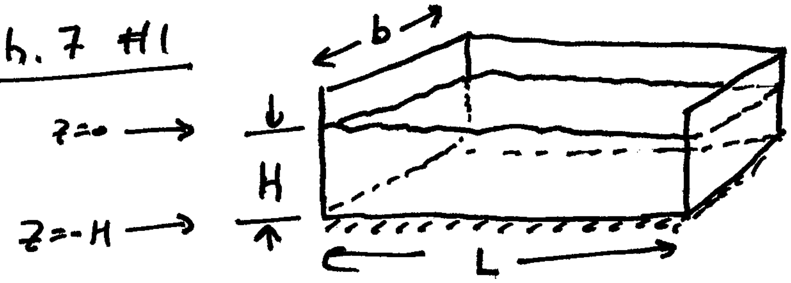
HW 4 APPH 4200 PHYSICS OF FLUIDS NOV 8, 2007

SOLUTIONS

Ch. 7 #1



$\bar{u} = \nabla \varphi$

$\nabla^2 \varphi = 0$

$$\varphi(x,y,z,t) = A\cos\left(\frac{m\pi x}{L}\right) \otimes \cos\left(\frac{n\pi y}{b}\right)\cosh(k(z+H))e^{-i\omega t}$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$= \left[-\left(\frac{m\pi}{L}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + k^2\right]\varphi \quad \text{so} \quad k^2 = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

BOUNDARY CONDITIONS AT SURFACE ($z=0$) IS:

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t} \quad \text{AND} \quad \frac{\partial\varphi}{\partial t} = -g\eta \quad \text{OR} \quad \frac{\partial^2\varphi}{\partial t^2} = -g\frac{\partial\varphi}{\partial z} \quad \text{at } z=0$$

$$\frac{\partial^2\varphi}{\partial t^2} \propto -\omega^2\cosh(kH) \qquad \frac{\partial\varphi}{\partial z} \propto k\sinh(kH)$$

$$\therefore \quad \omega^2\cosh(kH) = +gk\sinh(kH) \qquad \text{Q.E.D.}$$

Ch. 7 #2

$k^2 = \left(\frac{\pi}{L}\right)^2$ WHEN $(m,n) = (1,0)$

THEN $\omega^2 = \frac{g\pi}{L}\tanh\left(\frac{\pi H}{L}\right)$ $\qquad \frac{g}{L} = \frac{9.8}{3\times10^4}\ s^{-2}$

$\frac{H}{L} = \frac{10^2}{3\times10^4}$

$\omega = 0.00328$ RAD/SEC

$T = \frac{2\pi}{\omega} = 1917$ SEC $= 32$ MIN

NOTE: THESE ARE THE RESONANT FREQUENCIES FOR A <u>LONG</u> CHANNEL (LIKE A RIVER OR CANAL)

Ch 7 H4

$$\omega^2 = k\left(g + \frac{\sigma k^2}{\rho}\right)\tanh(kH) \quad \text{As } kH \to \infty,\ \tanh(kH) \to 1$$

So $$\omega^2 = kg\left(1 + \tfrac{1}{2}a^2k^2\right) \quad \text{with} \quad a^2 = \frac{2\sigma}{\rho g}$$

But $c_g = \dfrac{\partial \omega}{\partial k}$ so...

$$2\omega \frac{\partial \omega}{\partial k} = g\left(1 + \tfrac{1}{2}a^2k^2\right) + kg(a^2k)$$

$$\frac{\partial \omega}{\partial k} = \frac{g}{2\omega}\left(1 + \tfrac{3}{2}a^2k^2\right) = \frac{1}{2}\sqrt{\frac{g}{k}}\,\frac{\left(1 + \frac{3}{2}a^2k^2\right)}{\sqrt{1 + \frac{1}{2}a^2k^2}}$$

Since $k = \dfrac{2\pi}{\lambda}$, this can be rewritten ...

$$\frac{\partial \omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{2\pi}\lambda}\,\frac{\left(1 + 6\pi^2\, a^2/\lambda^2\right)}{\sqrt{1 + 2\pi^2\, a^2/\lambda^2}}$$

As $\dfrac{a}{\lambda} \to 0$, $\dfrac{\partial \omega}{\partial k} = \dfrac{1}{2}\sqrt{\dfrac{g}{2\pi}\lambda}$

$\dfrac{a}{\lambda} \to \infty$ $\quad \dfrac{\partial \omega}{\partial k} = \dfrac{3}{2}\sqrt{\dfrac{g a^2}{2\pi\lambda}}$

Where is the <u>minimum</u> group velocity?

When $$\frac{\partial^2 \omega}{\partial k^2} = \frac{\partial^2 \omega}{\partial \lambda^2} = 0$$

But $$\frac{\partial^2 \omega}{\partial k^2} = \frac{\partial}{\partial k}\left[\frac{g}{2\omega}\left(1 + \tfrac{3}{2}a^2k^2\right)\right] = 0$$

$$= -\frac{g}{\omega^2}\frac{\partial \omega}{\partial k}\left(1 + \tfrac{3}{2}a^2k^2\right) + \frac{3g}{\omega}a^2k = 0$$

Or $$\frac{\partial \omega}{\partial k}\left(1 + \tfrac{3}{2}a^2k^2\right) = \left(\frac{\omega}{k}\right)3a^2k^2$$

$$\frac{g}{2\omega}\left(1 + \tfrac{3}{2}a^2k^2\right)^2 = \left(\frac{k}{\omega}\right)3a^2k^2\left(\frac{g}{k}\right)\left(1 + \tfrac{1}{2}a^2k^2\right)$$

(#4 CONT.)

ELIMINATING TERMS ...

$$\frac{1}{2}\left(1+\frac{3}{2}a^2k^2\right)^2 = 3a^2k^2\left(1+\frac{1}{2}a^2k^2\right)$$

THIS IS A QUADRATIC EQUATION FOR (a^2k^2).

SOLVING ...

$$\frac{1}{2} - \frac{3}{2}a^2k^2 - \frac{3}{8}(a^2k^2)^2 = 0$$

OR $$(a^2k^2)^2 + 4a^2k^2 - \frac{4}{3} = 0$$

$$(a^2k^2) = -2 \pm \sqrt{16/3}$$ ONLY POSITIVE ROOT IS PHYSICAL

GIVING $$a^2k^2 = 2\left(\frac{2}{\sqrt{3}} - 1\right) = \frac{2\sigma}{\rho g}k^2$$ Q.E.D.