APPH 4200 PHYSICS OF FLUIDS
HW3 SOLUTIONS

#1) THE VORTICITY EQUATION IS THE CURL OF THE NAVIER-STOKES EQ.. FOR INCOMPRESSIBLE, CONSTANT DENSITY FLUID, THIS IS...

$$\nabla \times \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \bar{u} \right)$$

$$\bar{\Omega} \equiv \nabla \times \bar{u}$$

$$\bar{u} \cdot \bar{\nabla}\, \bar{u} = \bar{\nabla}\left(\tfrac{1}{2} u^2\right) - \bar{u} \times \bar{\Omega}$$

SO

$$\frac{\partial \bar{\Omega}}{\partial t} + \nabla \times (\bar{\Omega} \times \bar{u}) = \nu \nabla^2 \bar{\Omega}$$

BUT

$$\nabla \times (\bar{\Omega} \times \bar{u}) = \bar{\Omega}\, \nabla \cdot \bar{u} - \bar{u}\, \bar{\nabla} \cdot \bar{\Omega} + \bar{u} \cdot \bar{\nabla} \bar{\Omega} - (\bar{\Omega} \cdot \bar{\nabla}) \bar{u}$$

SO

$$\frac{\partial \bar{\Omega}}{\partial t} + (\bar{u} \cdot \bar{\nabla}) \bar{\Omega} = \underbrace{(\bar{\Omega} \cdot \bar{\nabla}) \bar{u}}_{\text{VORTEX STRETCHING}} + \underbrace{\nu \nabla^2 \bar{\Omega}}_{\text{VORTEX DISSIPATION}}$$

FOR TWO-DIMENSIONAL FLOW $(\bar{\Omega} \cdot \bar{\nabla})\, \bar{u} = 0$ ($\bar{\Omega}$ IS ALONG AXIS OF SYMMETRY!).

FOR AXI-SYMMETRIC FLOW, $(\bar{u} \cdot \bar{\nabla})\, \bar{\Omega} = 0$.

THEREFORE,

$$\frac{\partial \bar{\Omega}}{\partial t} = \nu \nabla^2 \bar{\Omega}$$

IN CYLINDRICAL COORDINATES (APPENDIX B), WE HAVE

$$\frac{\partial \Omega}{\partial t} = \frac{\nu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Omega}{\partial r} \right)$$

#1 (CONT.) SUBSTITUTE $\Omega = \frac{\Phi}{4\pi\nu t} e^{-r^2/4\nu t}$

$$\frac{\partial \Omega}{\partial t} = -\frac{\Omega}{t} + \frac{r^2}{4\nu t^2}\Omega = -\frac{\Omega}{t}\left(1 - \frac{r^2}{4\nu t}\right)$$

$$\frac{\partial \Omega}{\partial r} = -\frac{r}{2\nu t}\Omega$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial \Omega}{\partial r}\right) = -\frac{r}{\nu t}\Omega + \frac{r^2}{4\nu^2 t^2}\Omega = -\frac{\Omega}{\nu t}\left(1 - \frac{r^2}{4\nu t}\right)$$

THUS Ω SATISFIES VORTICITY EQUATION

TO FIND VELOCITY DISTRIBUTION....

$$\Omega(r) = \frac{1}{r}\frac{\partial}{\partial r}(r U_\theta) \qquad \text{(APPENDIX B)}$$

INTEGRATING...

$$r U_\theta(r) = \int_0^r dr\, r e^{-r^2/4\nu t} \times \frac{\Phi}{4\pi\nu t}$$

IF $x \equiv r^2/4\nu t$, THEN $r\,dr = 2\nu t\,dx$, AND

$$r U_\theta = \frac{\Phi}{2\pi}\left(1 - e^{-r^2/4\nu t}\right)$$

THE CIRCULATION FOR A LINE VORTEX IS

$$r U_\theta = \frac{\Gamma}{2\pi}$$

SO $\Phi = \Gamma$. !

FOR A FREE, UNDRIVEN LINE VORTEX, THE VORTICITY DIFFUSES AND DECAYS TO ZERO WITH TIME. EVEN THOUGH VISCOUS FORCE VANISHES... VISCOUS DISSIPATION DOES NOT VANISH FOR LINE VORTEX.

#2] $\bar{u} = (u_r, u_\theta, u_z) = (0, arz, 0)$

$\bar{\Omega} = (\Omega_r, \Omega_\theta, \Omega_z)$ (USE APPENDIX B)

a) $\Omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = -ar$

$\Omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0$

$\Omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(a r^2 z) = 2az$

b) $\nabla \cdot \bar{\Omega} = \frac{1}{r}\frac{\partial}{\partial r}(r\Omega_r) + \frac{1}{r}\frac{\partial}{\partial \theta} u_\theta + \frac{\partial u_z}{\partial z}$

$= -2a + 2a = 0$

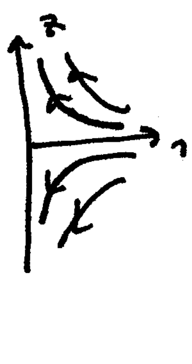
c) EQUATIONS FOR STREAMLINES ARE CIRCLES ABOUT AXIS OF SYMMETRY

VORTEX LINES: $\frac{dr}{\Omega_r} = \frac{dz}{\Omega_z}$

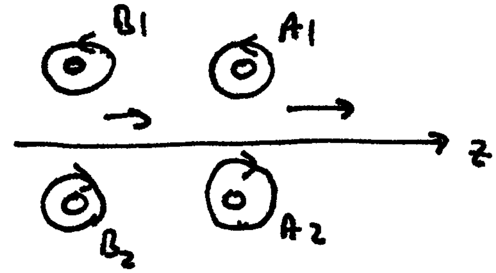
$-\frac{dr}{ar} = \frac{dz}{2az}$

OR $\frac{1}{2}\ln z + \ln r = \text{CONSTANT}$

$\ln(r z^2) = \text{CONSTANT}$ OR $r z^2 = \text{CONSTANT}$

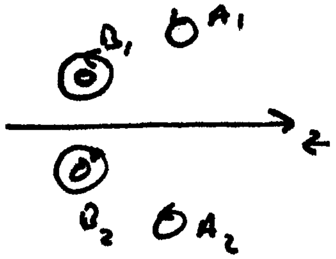


#3 INITIAL:

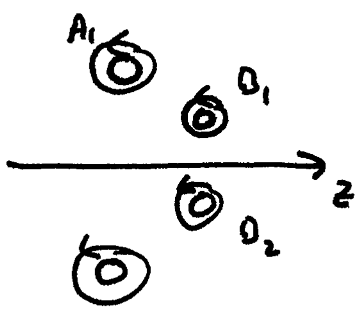


WHAT HAPPENS:

- A1-A2 AND B1-B2 CAUSE VORTEX RINGS TO TRANSLATE.
- B1→A1, B2→A2 CAUSE VORTEX-A TO INCREASE DIAMETER
- A1→B1, A2→B2 CAUSE VORTEX-B TO DECREASE DIAMETER.



- AFTER DIAMETER CHANGES, B1-A1 AND B2-A2 CAUSE VORTEX-A TO SLOW AND VORTEX-B TO ACCELERATE.



- AFTER VORTEX-B PASSES THROUGH A, THEN MUTUAL INTERACTION CAUSES VORTEX-B TO INCREASE DIAMETER AND VORTEX-A TO DECREASE DIAMETER.
- LEAP-FROG CONTINUES/REPEATS

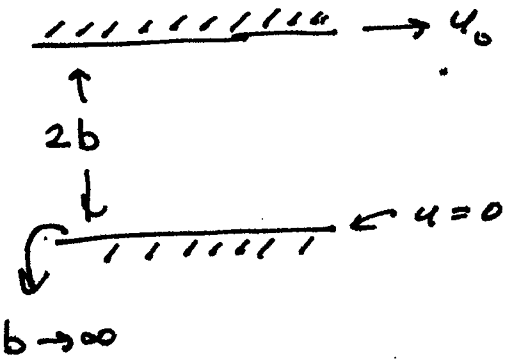
#4 SECTION 9.7 IN TEXT BOOK DISCUSSES THE "IMPULSIVELY" STARTED PLATE. Fig. 9.9 SHOWS THE EVOLUTION OF THE FLOW PROFILE. THE CHARACTERISTIC TIME TO RELAX TO STEADY COUETTE FLOW IS $\tau \sim b^2/\nu$.

IN CLASS, WE SOLVED THIS PROBLEM USING FOURIER'S METHOD. HERE, WE DISCUSS THE TEXT.

#4 CONT. NOTE! ANY QUALITATIVE DESCRIPTION OF VELOCITY DIFFUSION DUE TO VISCOSITY IS AN ACCEPTABLE ANSWER TO THIS QUESTION

IN SEC 9.7, THE TEXT CONSIDERS DIFFUSION FROM A WALL WITH THE OPPOSITE WALL OFF AT A VERY LARGE SEPARATION DISTANCE, i.e. $b \to \infty$

THEN, WE NEED DISCUSS DIFFUSION ONLY FROM TOP PLATE.



SIMILARITY SOLUTION

$$u(y,t) = u_0 F(\eta)$$

$$\eta \equiv y/2\sqrt{\nu t}$$

AT $t=0$, $\eta = \infty$ AND $F(\infty)=0$

at $t=\infty$, $\eta = 0$ AND $F(0)=1$

NEXT, $\dfrac{\partial u}{\partial t} = u_0 F' \dfrac{\partial \eta}{\partial t} \quad \left(\dfrac{\partial \eta}{\partial t} = -\dfrac{1}{2}\dfrac{\eta}{t}\right)$

$$\frac{\partial u}{\partial y} = u_0 F' \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = u_0 F'' \left(\frac{\partial \eta}{\partial y}\right)^2 \quad \left(\frac{\partial \eta}{\partial y} = \frac{1}{2\sqrt{\nu t}}\right)$$

SO $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{-2\eta F' = F''}$

THIS IS AN ORDINARY DIFFERENTIAL EQUATION! MUCH EASIER THAN A PDE.

INTEGRATING ONCE: $\dfrac{\partial F}{\partial \eta} = C_1 e^{-\eta^2}$

INTEGRATING TWICE: $F(\eta) - F(0) = C_1 \int_0^\eta d\eta\, e^{-\eta^2}$

OR $F(\eta) = 1 - \dfrac{2}{\sqrt{\pi}} \int_0^\eta d\eta\, e^{-\eta^2}$.