

4200 HW #1 SOLUTIONS

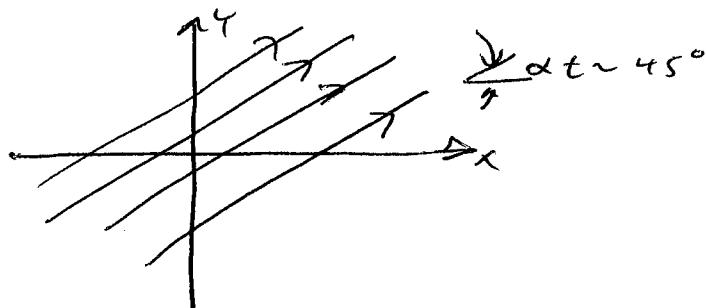
PROBLEM #1

STREAMLINES

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{OR} \quad \frac{dx}{\cos(\alpha t)} = \frac{dy}{\sin(\alpha t)}$$

$$\frac{dy}{dx} = \frac{\sin \alpha t}{\cos \alpha t} = \tan \alpha t$$

For any given time, αt is the angle of the streamlines which are straight lines.



TRAJECTORY (or PATHLINES) OF A PARTICLE RELEASED AT $t=0$ BEGINS AT $(x, y) = (0, 0)$.

$$\frac{dx}{dt} = \cos \alpha t \quad \frac{dy}{dt} = \sin \alpha t$$

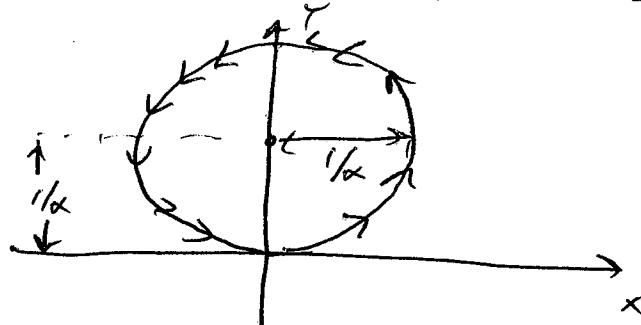
$$\text{so } x(t) = C_x + \frac{1}{\alpha} \sin(\alpha t) \quad y(t) = C_y - \frac{1}{\alpha} \cos(\alpha t)$$

$$\text{initial condition } C_x = 0 \quad C_y = \frac{1}{\alpha}$$

$$x(t) = \frac{1}{\alpha} \sin(\alpha t) \quad y(t) = \frac{1}{\alpha} (1 - \cos(\alpha t))$$

$$\text{note: } x^2 + \left(y - \frac{1}{\alpha}\right)^2 = \frac{1}{\alpha^2} \quad \text{so trajectory}$$

are circles!



PROBLEM #1 (CONTINUED)

(2)

STREAKLINES FROM SMOKE RELEASED CONTINUOUSLY

AT $(x, y) = (0, 0)$. THE STREAKLINE IS JUST A TRAJECTORY EXCEPT $(x(t'), y(t')) = (0, 0)$, WHERE t' = TIME WHEN SMOKE IS RELEASED.

SOLUTION IS

$$x(t) = C_x(t') + \frac{1}{\alpha} \sin(\alpha t) \quad y(t) = C_y(t') - \frac{1}{\alpha} \cos(\alpha t)$$

"initial condition"

$$(x(t'), y(t')) = (0, 0) \quad \text{OR}$$

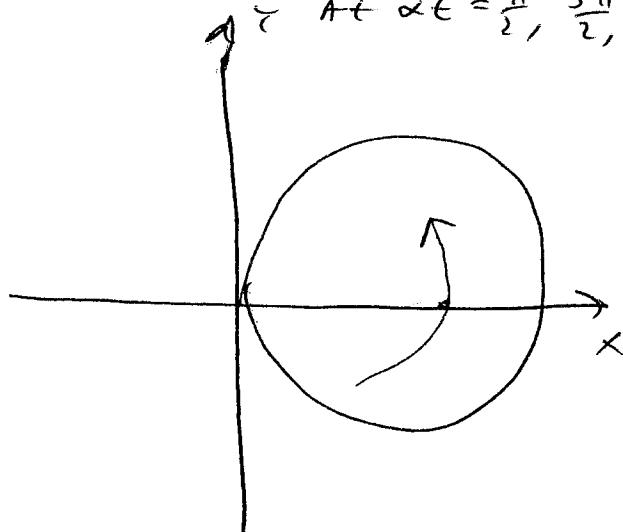
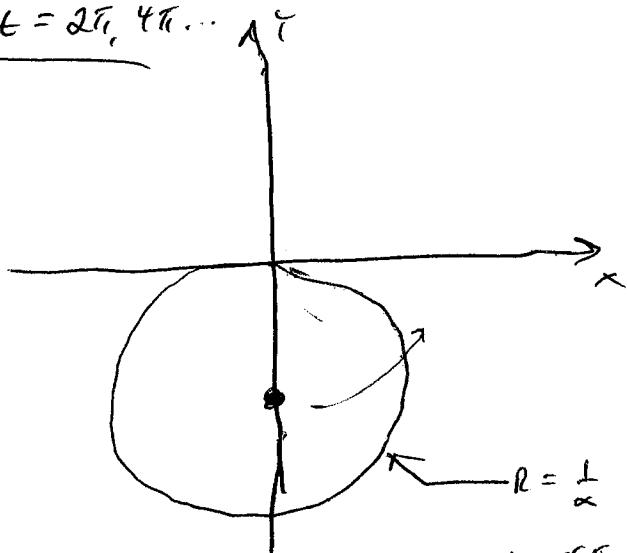
$$x(t) = \frac{1}{\alpha} (\sin(\alpha t) - \sin(\alpha t')) \quad y(t) = -\frac{1}{\alpha} (\cos(\alpha t) - \cos(\alpha t'))$$

WE'LL LIKE AN EQUATION FOR THE ENTIRE STREAKLINE AT A TIME $t > t'$ THAT IS INDEPENDENT OF t' . ELIMINATING t' BY COMPUTING THE SQUARE...

$$(x - \frac{1}{\alpha} \sin(\alpha t))^2 + (y + \frac{1}{\alpha} \cos(\alpha t))^2 = \frac{1}{\alpha^2}$$

$$y \text{ at } \alpha t = \frac{\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\text{at } \alpha t = 2\pi, 4\pi, \dots$$



THESE ARE CIRCLES THAT ROTATE ABOUT THE ORIGIN:

$$(x, y) = (\sin(\alpha t), -\cos(\alpha t)) / \alpha$$

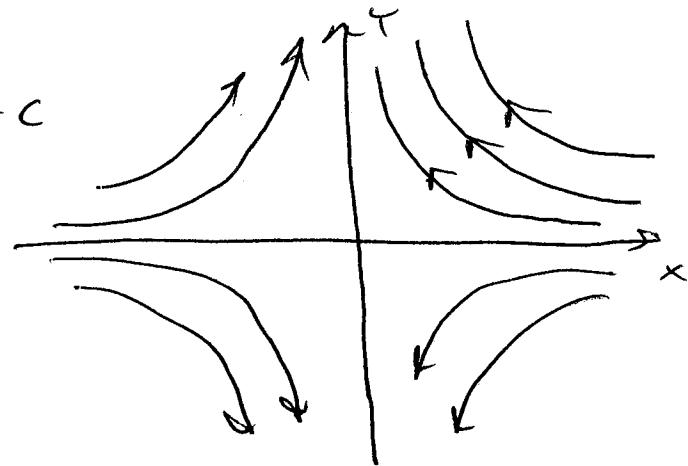
PROBLEM #2

(3)

STREAMLINES

$$-\frac{dx}{x} = \frac{dy}{y} \quad \text{OR} \quad -\ln x = \ln y + C$$

$$xy = C$$



STRAIN AND ROTATION

$$\frac{\partial u_i}{\partial x_j} = \epsilon_{ij} + \gamma_{ij}/2$$

$$= \begin{Bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} \end{Bmatrix} = \begin{Bmatrix} -1 & 0 \\ 0 & 1 \end{Bmatrix} \leftarrow \text{THIS IS SYMMETRIC!}$$

$$\text{so } \epsilon_{ij} = \begin{Bmatrix} -1 & 0 \\ 0 & 1 \end{Bmatrix} \quad \gamma_{ij} = 0 \quad \text{since } \gamma_{ij} = \text{Anti-Symmetric part}$$

NOTE: FLOW IS IRROTATIONAL.

TRACER FIELD FROM CONTINUOUS SOURCE

IN STEADY STATE, ...

$$\bar{u} \cdot \bar{\nabla} c(x, y, t) = s(x)$$

$$-x \frac{\partial c}{\partial x} + y \frac{\partial c}{\partial y} = x e^{-x}$$

SINCE s DEPENDS ONLY UPON x , SEEK SOLUTIONS FOR c THAT ONLY DEPENDS ON x . THUS,

$$\frac{\partial c}{\partial x} = -e^{-x}$$

$$c(x, y) \sim d + e^{-x}$$

WHERE DOES TRACER GO?

TRACER SOURCE $s(x)$ IS ENTIRELY SWEEPED UP BY FLOW TO $y \rightarrow \pm\infty$

d = A CONSTANT OR ANY FUNCTION OF x . SINCE A FUNCTION OF x IS CONSTANT ALONG STREAMLINES

PROBLEM #3

$$\begin{aligned}
 ((\mathbf{v} \times \mathbf{u}) \times \mathbf{u})_i &= \epsilon_{ijk} (\mathbf{v} \times \mathbf{u})_j u_k \\
 &= \epsilon_{ijk} \epsilon_{ilm} u_k \frac{\partial u_m}{\partial x_l} = - \epsilon_{ikj} \epsilon_{ilm} u_k \frac{\partial u_m}{\partial x_l} \\
 &= - (\delta_{il} \delta_{km} - \delta_{im} \delta_{lk}) u_k \frac{\partial u_m}{\partial x_l} \\
 &= - \frac{\partial u_k}{\partial x_i} u_k + u_k \frac{\partial u_i}{\partial x_k} \\
 &= - \frac{2}{2x_i} \left(\frac{1}{2} u^2 \right) + u_k \frac{\partial u_i}{\partial x_k} \quad Q.E.D.
 \end{aligned}$$

PROBLEM #4 (ch 3 PG 6)

PATH LINES $\frac{dx}{dt} = \frac{x}{1+t}$ $\frac{dy}{dt} = \frac{2y}{2+t}$

$$x(t) = C_x (1+t) \quad y(t) = C_y (2+t)^2$$

AT $t=0$ $(x, y) = (x_0, y_0)$, so THAT

$$x(t) = x_0 (1+t) \quad \text{and} \quad y(t) = \frac{y_0}{4} (2+t)^2$$

STREAMLINES

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{x} (1+t) = \frac{dy}{2y} (2+t) \quad \text{FOR FIXED } y$$

INTEGRATE ...

\curvearrowright A CONSTANT OR A FUNCTION OF TIME

$$(1+t) \ln x = 1/2 (2+t) \ln y + C$$

$$y^{1/2(2+t)} = C x^{(1+t)}$$

$$y = C x^{\frac{2(1+t)}{(1+t)}}$$

$$\begin{cases} \text{as } t \rightarrow 0 \quad y = C x \\ \text{as } t \rightarrow \infty \quad y = C x^2 \end{cases}$$

THIS EXPRESSION FOR STREAMLINES
IS VALID FOR $x > 0$ AND $y > 0$.

Problem #5 (Ch 4 Problem 2)

(5)

A MATERIAL VOLUME MEANS A VOLUME MOVING
WITH THE FLUID. FOR ANY FLUID QUANTITY,
WE HAVE

$$\frac{d}{dt} \iiint_V F dV = \iiint_V \frac{\partial F}{\partial t} dV + \iint_A F \bar{u} \cdot d\bar{A}$$

FLUID / BOUNDARY
VELOCITY

Using Gauss's Law

$$= \iiint \left(\frac{\partial F}{\partial t} + \nabla \cdot (F \bar{u}) \right) dV$$

Now, if $F = \rho$ (mass density), then at $\iiint_V F dV = 0$
since mass is not created or destroyed.

Hence $\frac{\partial F}{\partial t} + \nabla \cdot (F \bar{u}) = 0$ with $F = \rho$