$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,} \qquad (4.8)$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\mathbf{u} = 0.} \qquad (4.9)$$

$$\boxed{\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial\tau_{ij}}{\partial x_j}.} \qquad (4.15)$$

$$\boxed{\tau_{ij} = -\left(p + \tfrac{2}{3}\mu\nabla\cdot\mathbf{u}\right)\delta_{ij} + 2\mu e_{ij}} \qquad (4.43)$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left[2\mu e_{ij} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})\delta_{ij}\right], \qquad (4.44)$$

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \mu\ \nabla^2\mathbf{u}.} \quad \text{(incompressible)} \qquad (4.45)$$

$$\text{Deformation work (rate per volume)} = -p(\nabla\cdot\mathbf{u}) + \phi, \qquad (4.59)$$

where

$$\phi \equiv 2\mu e_{ij}e_{ij} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})^2 = 2\mu\left[e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})\delta_{ij}\right]^2. \qquad (4.60)$$

$$\rho\frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho g_i u_i + \frac{\partial}{\partial x_j}(u_i\tau_{ij}) - \tau_{ij}e_{ij}, \qquad (4.61)$$

$$\rho\frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \underbrace{\rho\mathbf{g}\cdot\mathbf{u}}_{\substack{\text{rate of work by}\\ \text{body force}}} + \underbrace{\frac{\partial}{\partial x_j}(u_i\tau_{ij})}_{\substack{\text{total rate of}\\ \text{work by } \boldsymbol{\tau}}} + \underbrace{p(\nabla\cdot\mathbf{u})}_{\substack{\text{rate of work}\\ \text{by volume}\\ \text{expansion}}} - \underbrace{\phi}_{\substack{\text{rate of}\\ \text{viscous}\\ \text{dissipation}}} \qquad (4.62)$$

Appendix A

Some Properties of Common Fluids

A1. Useful Conversion Factors

Length:	1 m = 3.2808 ft 1 in. = 2.540 cm 1 mile = 1.609 km 1 nautical mile = 1.852 km
Mass:	1 kg = 2.2046 lb 1 metric ton = 1000 kg
Time:	1 day = 86,400 s
Density:	$1\ \text{kg/m}^3 = 0.062428\ \text{lb/ft}^3$
Velocity:	1 knot = 0.5144 m/s
Force:	$1\ \text{N} = 10^5\ \text{dyn}$
Pressure:	$1\ \text{dyn/cm}^2 = 0.1\ \text{N/m}^2 = 0.1\ \text{Pa}$ $1\ \text{bar} = 10^5\ \text{Pa}$
Energy:	$1\ \text{J} = 10^7\ \text{erg} = 0.2389\ \text{cal}$ 1 cal = 4.186 J
Energy flux:	$1\ \text{W/m}^2 = 2.39 \times 10^{-5}\ \text{cal cm}^{-2}\ \text{s}^{-1}$

A2. Properties of Pure Water at Atmospheric Pressure

Here, ρ = density, α = coefficient of thermal expansion, μ = shear viscosity, ν = kinematic viscosity = μ/ρ, κ = thermal diffusivity = $k/(\rho C_p)$, [k is first defined on p.6] Pr = Prandtl number, and 1.0×10^{-n} is written as $1.0\text{E} - n$

T °C	ρ kg/m^3	α K^{-1}	μ kg m^{-1} s^{-1}	ν m^2/s	κ m^2/s	C_p J kg^{-1} K^{-1}	Pr ν/κ
0	1000	−0.6E − 4	1.787E − 3	1.787E − 6	1.33E − 7	4217	13.4
10	1000	+0.9E − 4	1.307E − 3	1.307E − 6	1.38E − 7	4192	9.5
20	997	2.1E − 4	1.002E − 3	1.005E − 6	1.42E − 7	4182	7.1
30	995	3.0E − 4	0.799E − 3	0.802E − 6	1.46E − 7	4178	5.5
40	992	3.8E − 4	0.653E − 3	0.658E − 6	1.52E − 7	4178	4.3
50	988	4.5E − 4	0.548E − 3	0.555E − 6	1.58E − 7	4180	3.5

Latent heat of vaporization at 100 °C = 2.257×10^6 J/kg.
Latent heat of melting of ice at 0 °C = 0.334×10^6 J/kg.
Density of ice = 920 kg/m^3.
Surface tension between water and air at 20 °C = 0.0728 N/m.
Sound speed at 25 °C $\simeq$ 1500 m/s.

A3. Properties of Dry Air at Atmospheric Pressure

T °C	ρ kg/m^3	μ kg m^{-1} s^{-1}	ν m^2/s	κ m^2/s	Pr ν/κ
0	1.293	1.71E − 5	1.33E − 5	1.84E − 5	0.72
10	1.247	1.76E − 5	1.41E − 5	1.96E − 5	0.72
20	1.200	1.81E − 5	1.50E − 5	2.08E − 5	0.72
30	1.165	1.86E − 5	1.60E − 5	2.25E − 5	0.71
40	1.127	1.87E − 5	1.66E − 5	2.38E − 5	0.71
60	1.060	1.97E − 5	1.86E − 5	2.65E − 5	0.71
80	1.000	2.07E − 5	2.07E − 5	2.99E − 5	0.70
100	0.946	2.17E − 5	2.29E − 5	3.28E − 5	0.70

At 20 °C and 1 atm,

$$C_p = 1012\,\text{J kg}^{-1}\,\text{K}^{-1}$$
$$C_v = 718\,\text{J kg}^{-1}\,\text{K}^{-1}$$
$$\gamma = 1.4$$
$$\alpha = 3.38 \times 10^{-3}\,\text{K}^{-1}$$
$$c = 340.6\,\text{m/s} \quad \text{(velocity of sound)}$$

Constants for dry air:

$$\text{Gas constant } R = 287.04\,\text{J kg}^{-1}\,\text{K}^{-1}$$
$$\text{Molecular mass } m = 28.966\,\text{kg/kmol}$$

A4. Properties of Standard Atmosphere

The following average values are accepted by international agreement. Here, z is the height above sea level.

z km	T °C	p kPa	ρ kg/m^3
0	15.0	101.3	1.225
0.5	11.5	95.5	1.168
1	8.5	89.9	1.112
2	2.0	79.5	1.007
3	−4.5	70.1	0.909
4	−11.0	61.6	0.819
5	−17.5	54.0	0.736
6	−24.0	47.2	0.660
8	−37.0	35.6	0.525
10	−50.0	26.4	0.413
12	−56.5	19.3	0.311
14	−56.5	14.1	0.226
16	−56.5	10.3	0.165
18	−56.5	7.5	0.120
20	−56.5	5.5	0.088

Appendix B

Curvilinear Coordinates

B1. Cylindrical Polar Coordinates

The coordinates are (R, θ, x), where θ is the azimuthal angle (see Figure 3.1b, where φ is used instead of θ). The equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_R u_R + \mathbf{i}_\theta u_\theta + \mathbf{i}_x u_x,$$

where $\mathbf{i}_R$, $\mathbf{i}_\theta$, and $\mathbf{i}_x$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla\psi = \mathbf{i}_R \frac{\partial\psi}{\partial R} + \frac{\mathbf{i}_\theta}{R}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_x \frac{\partial\psi}{\partial x}.$$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial x^2}.$$

Divergence of a vector

$$\nabla\cdot\mathbf{u} = \frac{1}{R}\frac{\partial(Ru_R)}{\partial R} + \frac{1}{R}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_x}{\partial x}.$$

Curl of a vector

$$\nabla\times\mathbf{u} = \mathbf{i}_R\left(\frac{1}{R}\frac{\partial u_x}{\partial\theta} - \frac{\partial u_\theta}{\partial x}\right) + \mathbf{i}_\theta\left(\frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R}\right) + \mathbf{i}_x\left[\frac{1}{R}\frac{\partial(Ru_\theta)}{\partial R} - \frac{1}{R}\frac{\partial u_R}{\partial\theta}\right].$$

Laplacian of a vector

$$\nabla^2\mathbf{u} = \mathbf{i}_R\left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\theta}{\partial\theta}\right) + \mathbf{i}_\theta\left(\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R^2}\right) + \mathbf{i}_x\nabla^2 u_x.$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$
\begin{aligned}
e_{RR} &= \frac{\partial u_R}{\partial R} = \frac{1}{2\mu}\sigma_{RR}, \\
e_{\theta\theta} &= \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R} = \frac{1}{2\mu}\sigma_{\theta\theta}, \\
e_{xx} &= \frac{\partial u_x}{\partial x} = \frac{1}{2\mu}\sigma_{xx}, \\
e_{R\theta} &= \frac{R}{2}\frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right) + \frac{1}{2R}\frac{\partial u_R}{\partial \theta} = \frac{1}{2\mu}\sigma_{R\theta}, \\
e_{\theta x} &= \frac{1}{2R}\frac{\partial u_x}{\partial \theta} + \frac{1}{2}\frac{\partial u_\theta}{\partial x} = \frac{1}{2\mu}\sigma_{\theta x}, \\
e_{xR} &= \frac{1}{2}\frac{\partial u_R}{\partial x} + \frac{1}{2}\frac{\partial u_x}{\partial R} = \frac{1}{2\mu}\sigma_{xR}.
\end{aligned}
$$

Vorticity ($\boldsymbol{\omega} = \nabla \times \mathbf{u}$)

$$
\begin{aligned}
\omega_R &= \frac{1}{R}\frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x}, \\
\omega_\theta &= \frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R}, \\
\omega_x &= \frac{1}{R}\frac{\partial}{\partial R}(R u_\theta) - \frac{1}{R}\frac{\partial u_R}{\partial \theta}.
\end{aligned}
$$

Equation of continuity

$$
\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(\rho R u_R) + \frac{1}{R}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial x}(\rho u_x) = 0.
$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$
\begin{aligned}
\frac{\partial u_R}{\partial t} + (\mathbf{u}\cdot\nabla)u_R - \frac{u_\theta^2}{R} &= -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\theta}{\partial \theta}\right), \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_R u_\theta}{R} &= -\frac{1}{\rho R}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2}\right), \\
\frac{\partial u_x}{\partial t} + (\mathbf{u}\cdot\nabla)u_x &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{u}\cdot\nabla &= u_R\frac{\partial}{\partial R} + \frac{u_\theta}{R}\frac{\partial}{\partial \theta} + u_x\frac{\partial}{\partial x}, \\
\nabla^2 &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}.
\end{aligned}
$$

B2. Plane Polar Coordinates

The plane polar coordinates are (r, θ), where r is the distance from the origin (Figure 3.1a). The equations for plane polar coordinates can be obtained from those of the cylindrical coordinates presented in Section B1, replacing R by r and suppressing all components and derivatives in the axial direction x. Some of the expressions are repeated here because of their frequent occurrence.

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$e_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr},$$
$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta},$$
$$e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu}\sigma_{r\theta}.$$

Vorticity

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) = 0.$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$
$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right),$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

B3. Spherical Polar Coordinates

The spherical polar coordinates used are (r, θ, φ), where φ is the azimuthal angle (Figure 3.1c). Equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_r u_r + \mathbf{i}_\theta u_\theta + \mathbf{i}_\varphi u_\varphi,$$

where $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\varphi$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla\psi = \mathbf{i}_r \frac{\partial\psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_\varphi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}.$$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

Divergence of a vector

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}.$$

Curl of a vector

$$\nabla\times\mathbf{u} = \frac{\mathbf{i}_r}{r\sin\theta}\left[\frac{\partial(u_\varphi\sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\varphi}\right] + \frac{\mathbf{i}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right] + \frac{\mathbf{i}_\varphi}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right].$$

Laplacian of a vector

$$\begin{aligned}\nabla^2\mathbf{u} = \mathbf{i}_r &\left[\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\\ &+ \mathbf{i}_\theta\left[\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right]\\ &+ \mathbf{i}_\varphi\left[\nabla^2u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right].\end{aligned}$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned}e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr},\\ e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta},\\ e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} = \frac{1}{2\mu}\sigma_{\varphi\varphi},\\ e_{\theta\varphi} &= \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} = \frac{1}{2\mu}\sigma_{\theta\varphi},\\ e_{\varphi r} &= \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = \frac{1}{2\mu}\sigma_{\varphi r},\\ e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta} = \frac{1}{2\mu}\sigma_{r\theta}.\end{aligned}$$

Vorticity

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\varphi \sin\theta) - \frac{\partial u_\theta}{\partial\varphi}\right],$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right],$$

$$\omega_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$

Equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho u_\varphi) = 0.$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2\cot\theta}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi\cot\theta}{r}$$
$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left[\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right],$$

where

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$