Bernard Thermal Instability: Marginal Mode Analysis

APPH 4200 Physics of Fluids
Columbia University
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Introduction

This notebook solves for the marginal instability threshold for Bernard thermal instability. See Chapter 12, Section 3 of Kundu and Cohen's textbook.

Four steps are required. First, from the linearized equations for the fluid dynamics, a cubic equation is solved for the vertical variation of the vertical velocity. Second, these three solutions are used to satisfy three boundary conditions at the constant-temperature walls. This leads to a condition between Ra (the Rayleigh number) and k^2 for marginal instability. Finally, we plot this condition and examine the modes.

Vertical Velocity Equation

The equation for vertical velocity takes the form

In[1]:= `eqVert = (q^2 + k^2)^3 == k^2 Ra`

Out[1]= $(k^2 + q^2)^3 == k^2\,\mathrm{Ra}$

when the vertical velocity has the form, $U_z \sim \mathrm{Cos}[q\,z]$.

In[2]:= `qSol = Solve[(eqVert /. q^2 → q2 /. k^2 → k2), q2]`

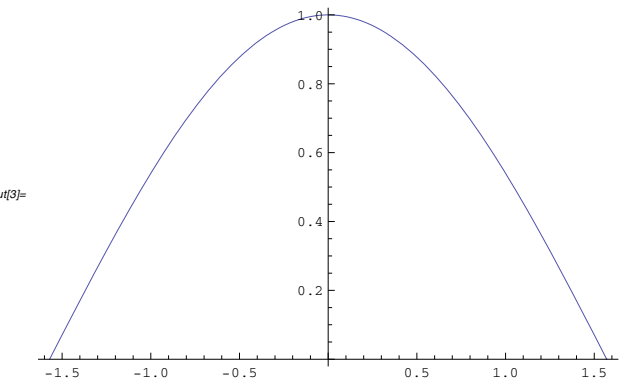
Out[2]= $\left\{\left\{q2 \to -k2 + k2^{1/3}\,\mathrm{Ra}^{1/3}\right\}, \left\{q2 \to -k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,\mathrm{Ra}^{1/3}\right\}, \left\{q2 \to -k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,\mathrm{Ra}^{1/3}\right\}\right\}$

Cos[z] is Even

First, we'll look for *even* modes...

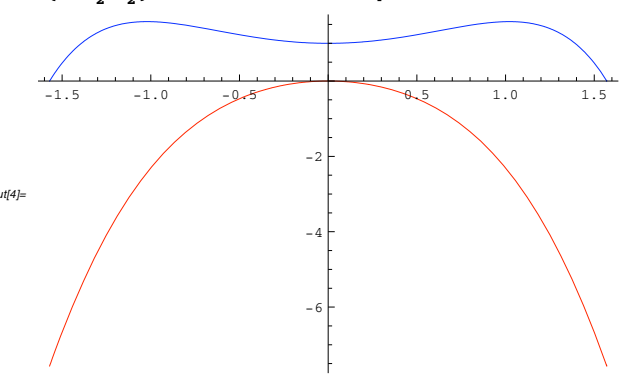
In[3]:= `Plot[Cos[z], {z, -π/2, π/2}]`

Out[3]=



In[4]:= `Plot[Evaluate[({Re[#1], Im[#1]} &)[Cos[z (1 + i √3)]]], {z, -π/2, π/2}, PlotStyle → {Blue, Red}]`

Out[4]=



An similiar analysis can be performed for the *odd* modes, but these modes have a higher critical Ra number.

Satisfy Boundary Conditions

In[5]:=
```
uz[z_] = a (Cos[Sqrt[q2] z] /. qSol[[1]]) +
   b (Cos[Sqrt[q2] z] /. qSol[[2]]) + c (Cos[Sqrt[q2] z] /. qSol[[3]])
```

Out[5]=
$$a \operatorname{Cos}\left[\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\; z\right] + b \operatorname{Cos}\left[\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\; z\right] + c \operatorname{Cos}\left[\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\; z\right]$$

In[6]:=
```
eq1 = uz[1/2] == 0
```

Out[6]=
$$a \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\right] + b \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right] + c \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right] == 0$$

In[7]:=
```
eq2 = (D[uz[z], z] == 0 /. z → 1/2)
```

Out[7]=
$$-a\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\; \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\right] - b\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\; \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right] - c\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\; \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right] == 0$$

In[8]:=
```
eq3 = (Collect[D[uz[z], {z, 4}] - 2 k2 D[uz[z], {z, 2}] + k2^2 uz[z],
   {a, b, c}, Simplify] == 0 /. z → 1/2)
```

Out[8]=
$$\frac{1}{2} i \left(i + \sqrt{3}\right) c\, k2^{2/3}\,Ra^{2/3} \operatorname{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] + a\, k2^{2/3}\,Ra^{2/3} \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\right] - \frac{1}{2} i \left(-i + \sqrt{3}\right) b\, k2^{2/3}\,Ra^{2/3} \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + \frac{1}{2} i \left(i + \sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right] == 0$$

These three equations can be satisfied simultaneous *only if* the determinant of a characteristic matrix vanishes. This defines the marginal condition between $k^2$and Ra.

Marginal Instability Condition

The marginal instability criterion is found by simultaneously solving three boundary conditions. This possible when the determinant of a characteristic matrix vanishes.

In[9]:=
```
First[eq1]
```

Out[9]=
$$a \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\right] + b \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right] + c \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right]$$

In[10]:=
```
row1 = {Coefficient[First[eq1], a],
   Coefficient[First[eq1], b], Coefficient[First[eq1], c]}
```

Out[10]=
$$\left\{\operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\right], \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right], \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right]\right\}$$

In[11]:=
```
row2 = {Coefficient[First[eq2], a],
   Coefficient[First[eq2], b], Coefficient[First[eq2], c]}
```

Out[11]=
$$\left\{-\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\; \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3}\,Ra^{1/3}}\right], -\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\; \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 - i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right], -\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\; \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 - \frac{1}{2}\left(1 + i\sqrt{3}\right) k2^{1/3}\,Ra^{1/3}}\right]\right\}$$

In[12]:=
```
row3 = {Coefficient[First[eq3], a],
  Coefficient[First[eq3], b], Coefficient[First[eq3], c]}
```

Out[12]=
$$\left\{k2^{2/3}\,Ra^{2/3}\,\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right],\ -\frac{1}{2}\,i\left(-i+\sqrt{3}\right)k2^{2/3}\,Ra^{2/3}\,\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+\frac{1}{2}\,i\left(i+\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right],\ \frac{1}{2}\,i\left(i+\sqrt{3}\right)k2^{2/3}\,Ra^{2/3}\,\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\right\}$$

In[13]:=
```
Det[{row1, row2, row3}] // Simplify
```

Out[13]=
$$\frac{1}{4}\,k2^{2/3}\,Ra^{2/3}\left(\sqrt{2}\left(-3-i\sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}\,\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]\mathrm{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]+\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\left(\sqrt{2}\left(3-i\sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}\,\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]\mathrm{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]+4\,i\sqrt{-3\,k2+3\,k2^{1/3}\,Ra^{1/3}}\,\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]\right)\right)$$

In[14]:=
```
marginal[k2_, Ra_] = (Det[{row1, row2, row3}] / k2^(2/3) / Ra^(2/3)) // Simplify
```

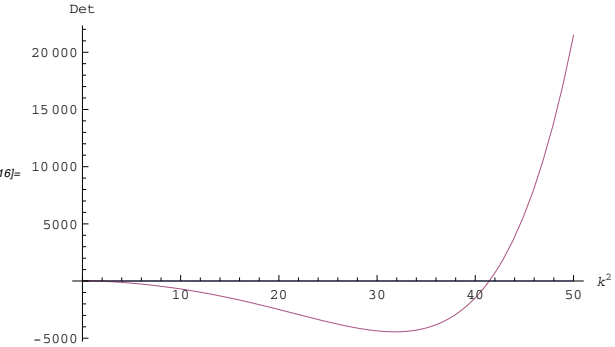
Out[14]=
$$\frac{1}{4}\left(\sqrt{2}\left(-3-i\sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}\,\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]\mathrm{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]+\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\left(\sqrt{2}\left(3-i\sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}\,\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]\mathrm{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]+4\,i\sqrt{-3\,k2+3\,k2^{1/3}\,Ra^{1/3}}\,\mathrm{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1-i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]\right)\right)$$

In[15]:=
```
marginal[2.0, 4000.0]
```

Out[15]= 0. - 14.6377 i

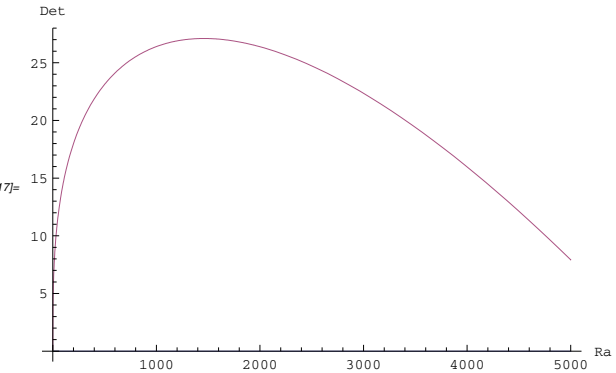
In[16]:=
```
Plot[Evaluate[({Re[#1], Im[#1]} &)[marginal[k2, 4000.`]]],
 {k2, 0, 50}, PlotRange → All,
 AxesLabel → {"\!\(\*SuperscriptBox[\(k\), \(2\)]\)", "Det"}]
```

Out[16]=



In[17]:=
```
Plot[Evaluate[({Re[#1], Im[#1]} &)[marginal[1.`, Ra]]],
 {Ra, 0, 5000}, PlotRange → All, AxesLabel → {"Ra", "Det"}]
```

Out[17]=



In[18]:=
```
FindRoot[marginal[1.0, Ra], {Ra, 2000.0}] // Chop
```

Out[18]= {Ra → 5854.48}

In[19]:=
```
FindRoot[marginal[5.0, Ra], {Ra, 2000.0}] // Chop
```

Out[19]= {Ra → 1967.8}

In[20]:=
```
FindRoot[marginal[50.0, Ra], {Ra, 2000.0}] // Chop
```
Out[20]= {Ra → 5048.62}

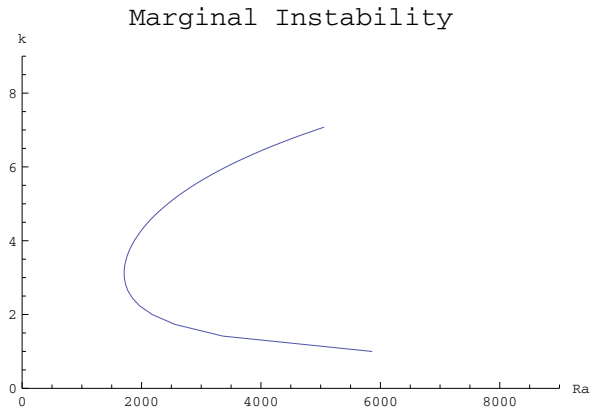
In[21]:=
```
marginalRa[k2_] := Ra /. Chop[FindRoot[marginal[k2, Ra], {Ra, 2000.0}]]
```

In[22]:=
```
marginalRa[2.0]
```
Out[22]= 3361.65

In[23]:=
```
marginalBoundary = Table[{marginalRa[k2], Sqrt[k2]}, {k2, 1.0, 50.0, 1.0}];
```

In[24]:=
```
evenMarginalPlt =
 ListPlot[marginalBoundary, Joined → True, PlotRange → {{0, 9000}, {0, 9}},
  PlotLabel → "Marginal Instability", AxesLabel → {"Ra", "k"}]
```

Out[24]=



Critical Mode

The critical mode and Rayleigh number were given in the textbook.

In[25]:=
```
critRa = Min[First[Transpose[marginalBoundary]]]
```
Out[25]= 1708.29

In[26]:=
```
critk = Select[marginalBoundary, (First[#] == critRa) &][[1, 2]]
```
Out[26]= 3.16228

In[27]:=
```
eq1Crit = eq1 /. k2 → critk^2 /. Ra → critRa /. a → 1 // Simplify
```
Out[27]= (3.33842 - 5.96343 i) b + (3.33842 + 5.96343 i) c == 0.402095

In[28]:=
```
eq2Crit = eq2 /. k2 → critk^2 /. Ra → critRa /. a → 1 // Simplify
```
Out[28]= (4.4566 - 38.593 i) b + (4.4566 + 38.593 i) c == 3.63419

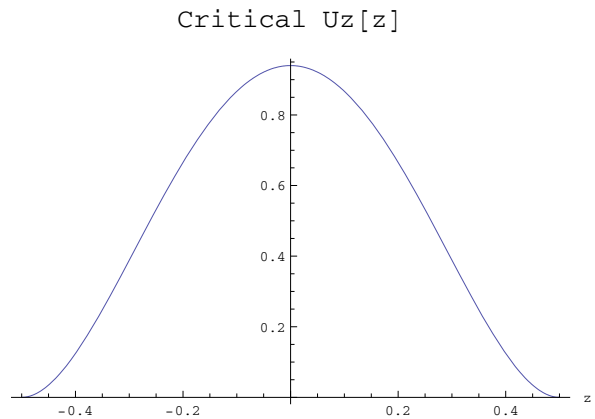
In[29]:=
```
modeCrit = First[Solve[{eq1Crit, eq2Crit}, {b, c}]]
```
Out[29]= {b → -0.03009 + 0.0505582 i, c → -0.03009 - 0.0505582 i}

In[30]:=
```
uzCrit[z_] = uz[z] /. a → 1 /. modeCrit /. k2 → critk^2 /. Ra → critRa
```
Out[30]= (-0.03009 + 0.0505582 i) Cos[(2.12995 + 5.23583 i) z] +
Cos[3.9692 z] - (0.03009 + 0.0505582 i) Cosh[(5.23583 + 2.12995 i) z]

In[31]:=
```
uzCrit[0.0]
```
Out[31]= $0.93982 - 1.38778 \times 10^{-17}$ i

In[32]:=
```
Plot[Evaluate[Chop[uzCrit[z]]], {z, -0.499`, 0.499`},
 PlotLabel → "Critical Uz[z]", AxesLabel → {"z", ""}]
```

Out[32]=



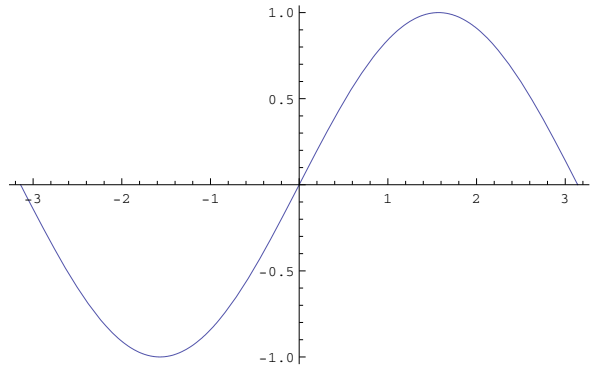
Odd Mode (?)

Sin[z] is Even

First, we'll look for *even* modes...

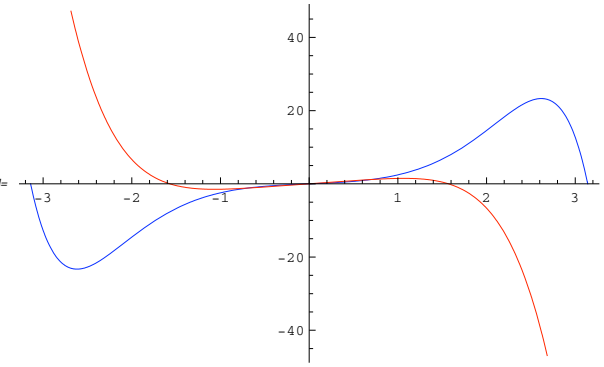
In[33]:=
```
Plot[Sin[z], {z, -π, π}]
```

Out[33]=



In[34]:=
```
Plot[Evaluate[({Re[#1], Im[#1]} &)[Sin[z (1 + i √3)]]],
  {z, -π, π}, PlotStyle → {Blue, Red}]
```

Out[34]=



An similiar analysis can be performed for the *odd* modes, but these modes have a higher critical Ra number.

Satisfy Boundary Conditions

In[35]:=
```
uz[z_] = a (Sin[Sqrt[q2] z] /. qSol[[1]]) +
   b (Sin[Sqrt[q2] z] /. qSol[[2]]) + c (Sin[Sqrt[q2] z] /. qSol[[3]])
```

Out[35]=
$$a\,\mathrm{Sin}\left[\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\;z\right]+b\,\mathrm{Sin}\left[\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\;z\right]+c\,\mathrm{Sin}\left[\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\;z\right]$$

In[36]:=
```
eq1 = uz[1 / 2] == 0
```

Out[36]=
$$a\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]+b\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]+c\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]==0$$

In[37]:=
```
eq2 = (D[uz[z], z] == 0 /. z → 1 / 2)
```

Out[37]=
$$a\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]+b\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]+c\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]==0$$

In[38]:=
```
eq3 = (Collect[D[uz[z], {z, 4}] - 2 k2 D[uz[z], {z, 2}] + k2^2 uz[z],
    {a, b, c}, Simplify] == 0 /. z → 1 / 2)
```

Out[38]=
$$\frac{1}{2}\,i\left(i+\sqrt{3}\right)c\,k2^{2/3}\,Ra^{2/3}\,\mathrm{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]+a\,k2^{2/3}\,Ra^{2/3}\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]-\frac{1}{2}\,i\left(-i+\sqrt{3}\right)b\,k2^{2/3}\,Ra^{2/3}\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+\frac{1}{2}\,i\left(i+\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]==0$$

These three equations can be satisfied simultaneous *only if* the determinant of a characteristic matrix vanishes. This defines the marginal condition between $k^2$and Ra.

■ Marginal Instability Condition

The marginal instability criterion is found by simultaneously solving three boundary conditions. This possible when the determinant of a characteristic matrix vanishes.

In[39]:=
```
First[eq1]
```

Out[39]=
$$a\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right]+b\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]+c\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]$$

In[40]:=
```
row1 = {Coefficient[First[eq1], a],
  Coefficient[First[eq1], b], Coefficient[First[eq1], c]}
```

Out[40]=
$$\left\{\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right],\ \mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right],\ \mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]\right\}$$

In[41]:=
```
row2 = {Coefficient[First[eq2], a],
  Coefficient[First[eq2], b], Coefficient[First[eq2], c]}
```

Out[41]=
$$\left\{\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right],\ \sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1-i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right],\ \sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{-k2-\frac{1}{2}\left(1+i\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right]\right\}$$

In[42]:=
```
row3 = {Coefficient[First[eq3], a],
  Coefficient[First[eq3], b], Coefficient[First[eq3], c]}
```

Out[42]=
$$\left\{k2^{2/3}\,Ra^{2/3}\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+k2^{1/3}\,Ra^{1/3}}\right],\ -\frac{1}{2}\,i\left(-i+\sqrt{3}\right)k2^{2/3}\,Ra^{2/3}\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{-k2+\frac{1}{2}\,i\left(i+\sqrt{3}\right)k2^{1/3}\,Ra^{1/3}}\right],\ \frac{1}{2}\,i\left(i+\sqrt{3}\right)k2^{2/3}\,Ra^{2/3}\,\mathrm{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\,k2^{2/3}+\left(1+i\sqrt{3}\right)Ra^{1/3}\right)}}{2\sqrt{2}}\right]\right\}$$

In[43]:= `Det[{row1, row2, row3}] // Simplify`

Out[43]=

$$\frac{1}{2\sqrt{2}} i \left(3 i + \sqrt{3}\right) k2^{2/3} \sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 - i\sqrt{3}\right) Ra^{1/3}\right)}\; Ra^{2/3} \operatorname{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 - i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3} Ra^{1/3}}\right] + \frac{1}{4} k2^{2/3} Ra^{2/3} \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 - i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \left(-4 i \sqrt{-3\, k2 + 3\, k2^{1/3} Ra^{1/3}} \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3} Ra^{1/3}}\right] \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] + \sqrt{2}\left(3 + i\sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)} \operatorname{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3} Ra^{1/3}}\right]\right)$$

In[44]:= `marginal[k2_, Ra_] = (Det[{row1, row2, row3}] / k2^(2/3) / Ra^(2/3)) // Simplify`

Out[44]=

$$\frac{1}{4}\left(-4 i \sqrt{-3\, k2 + 3\, k2^{1/3} Ra^{1/3}} \operatorname{Cos}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3} Ra^{1/3}}\right] \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 - i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] + \sqrt{2}\left(\left(3 + i\sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)} \operatorname{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] + i\left(3 i + \sqrt{3}\right)\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 - i\sqrt{3}\right) Ra^{1/3}\right)} \operatorname{Cos}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 - i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right] \operatorname{Sin}\left[\frac{\sqrt{-k2^{1/3}\left(2\, k2^{2/3} + \left(1 + i\sqrt{3}\right) Ra^{1/3}\right)}}{2\sqrt{2}}\right]\right) \operatorname{Sin}\left[\frac{1}{2}\sqrt{-k2 + k2^{1/3} Ra^{1/3}}\right]\right)$$

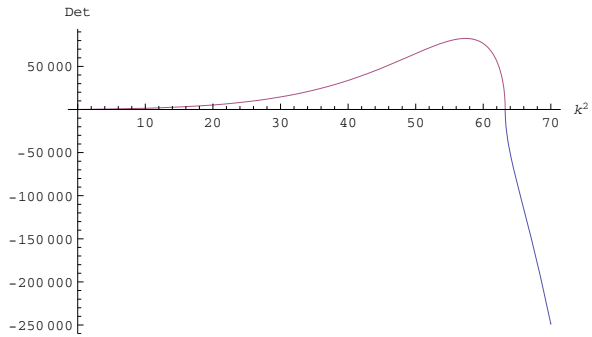
In[45]:=
```
marginal[50.0, 4000.0]
marginal[50.0, 1000.0]
```

Out[45]= 0. + 64 908.5 i

Out[46]= -49 386.4 + 0. i

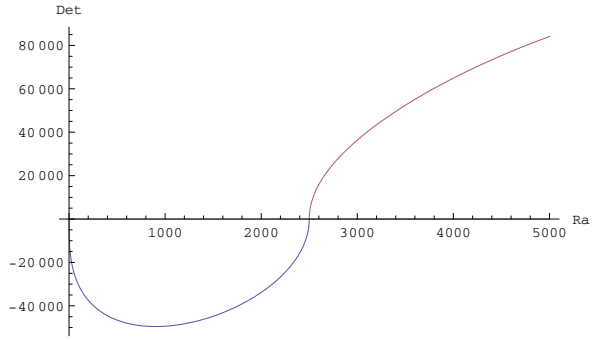
In[47]:=
```
Plot[Evaluate[({Re[#1], Im[#1]} &)[marginal[k2, 4000.`]]],
 {k2, 0, 70}, PlotRange → All,
 AxesLabel → {"\!\(\*SuperscriptBox[\(k\), \(2\)]\)", "Det"}]
```

Out[47]=



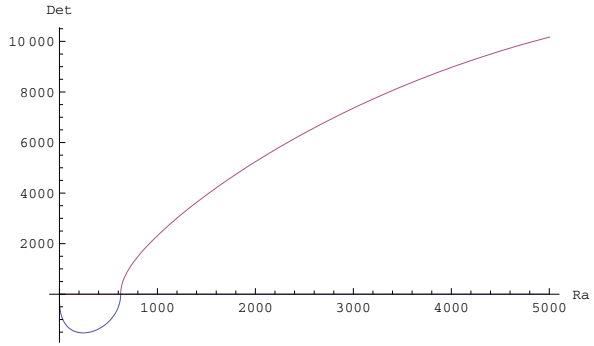
In[48]:=
```
Plot[Evaluate[({Re[#1], Im[#1]} &)[marginal[50.`, Ra]]],
 {Ra, 0, 5000}, PlotRange → All, AxesLabel → {"Ra", "Det"}]
```

Out[48]=



In[49]:=
```
Plot[Evaluate[({Re[#1], Im[#1]} &)[marginal[25.`, Ra]]],
 {Ra, 0, 5000}, PlotRange → All, AxesLabel → {"Ra", "Det"}]
```

Out[49]=



In[50]:=
```
FindRoot[marginal[25.0, Ra], {Ra, 3000.0}] // Chop
```

FindRoot::lstol :
The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances. ≫

Out[50]= {Ra → 625.}

In[51]:=
```
FindRoot[marginal[50.0, Ra], {Ra, 3000.0}] // Chop
```

FindRoot::lstol :
The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances. ≫

Out[51]= {Ra → 2500.}

In[52]:=
```
FindRoot[marginal[100.0, Ra], {Ra, 3000.0}] // Chop
```

Out[52]= {Ra → 32 104.1}

In[53]:=
```
marginalRa[k2_, start_: 3000.0] :=
 Ra /. Chop[FindRoot[marginal[k2, Ra], {Ra, start}]]
```

In[54]:=
```
globalStart = marginalRa[25.0, 500.0]
```

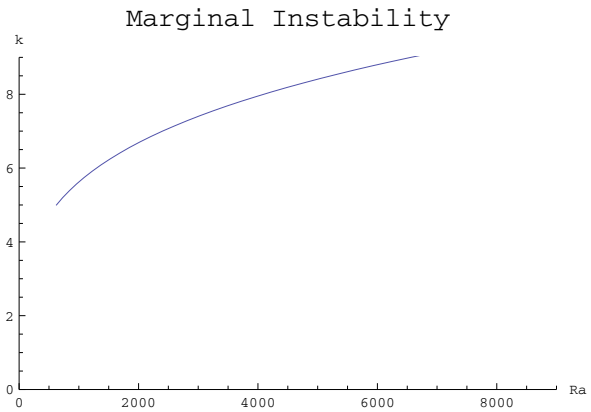
FindRoot::lstol :
The line search decreased the step size to within tolerance specified by AccuracyGoal and PrecisionGoal but was unable to find a sufficient decrease in the merit function. You may need more than MachinePrecision digits of working precision to meet these tolerances. ≫

Out[54]= 625.

In[55]:=
```
marginalBoundary =
  Table[{globalStart = marginalRa[k2, globalStart], Sqrt[k2]},
   {k2, 25.0, 100.0, 2.0}];
```

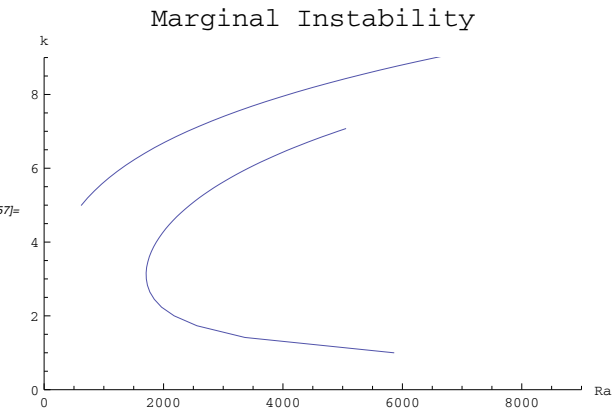
In[56]:=
```
oddMarginalPlt =
 ListPlot[marginalBoundary, Joined → True, PlotRange → {{0, 9000}, {0, 9}},
  PlotLabel → "Marginal Instability", AxesLabel → {"Ra", "k"}]
```

Out[56]=



In[57]:=
```
Show[evenMarginalPlt, oddMarginalPlt]
```

Out[57]=



■ Critical Mode

The critical mode and Rayleigh number were given in the textbook.

In[58]:=
```
critRa = Min[First[Transpose[marginalBoundary]]]
```

Out[58]= 625.

In[59]:=
```
critk = Select[marginalBoundary, (First[#] == critRa) &][[1, 2]]
```

Out[59]= 5.

In[60]:=
```
eq1Crit = eq1 /. k2 → critk^2 /. Ra → critRa /. a → 1 // Simplify
```

Out[60]= $\left(1.63254\times10^{-9}+1.87073\times10^{-9}\,i\right)+\left(1.-5.55112\times10^{-17}\,i\right)b+(0.135353-0.990797\,i)\,c==0$

In[61]:=
```
eq2Crit = eq2 /. k2 → critk^2 /. Ra → critRa /. a → 1 // Simplify
```

Out[61]= $\left(3.34603\times10^{-10}+6.76789\times10^{-10}\,i\right)+(1.+0.\,i)\,b+(0.607163-0.794577\,i)\,c==0$

In[62]:=
```
modeCrit = First[Solve[{eq1Crit, eq2Crit}, {b, c}]]
```

Out[62]= $\left\{b\to-3.24256\times10^{-9}+1.18201\times10^{-9}\,i,\ c\to3.24256\times10^{-9}+1.18201\times10^{-9}\,i\right\}$

In[63]:=
```
uzCrit[z_] = uz[z] /. a → 1 /. modeCrit /. k2 → critk^2 /. Ra → critRa
```

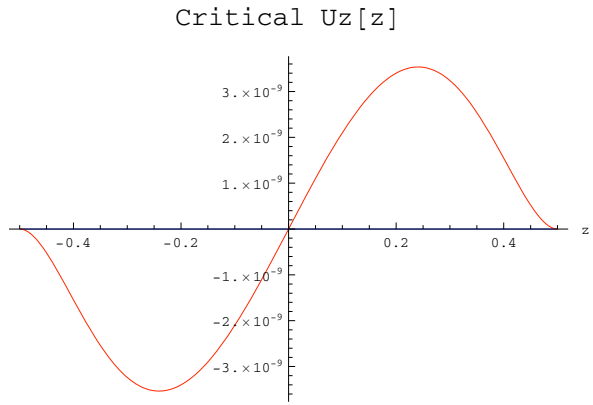
Out[63]=
$$\left(-3.24256\times10^{-9}+1.18201\times10^{-9}\,i\right)\mathrm{Sin}[(1.70313+6.35615\,i)\,z]+i\,\mathrm{Sinh}\left[\left(5.96046\times10^{-8}+0.\,i\right)z\right]+\left(1.18201\times10^{-9}-3.24256\times10^{-9}\,i\right)\mathrm{Sinh}[(6.35615+1.70313\,i)\,z]$$

In[64]:=
```
uzCrit[0.0]
```

Out[64]= 0. + 0. i

In[65]:=
```
Plot[Evaluate[{Re[uzCrit[z]], Im[uzCrit[z]]}],
 {z, -0.499`, 0.499`}, PlotLabel → "Critical Uz[z]",
 AxesLabel → {"z", ""}, PlotRange → All, PlotStyle → {Blue, Red}]
```

Out[65]=



Summary

The linearized fluid dynamics equations were solved to find the marginal instability boundary for Bernard thermal instability.