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Fluid viscosity and the attenuation of surface waves: a derivation based on conservation of energy

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Abstract
More than a century ago, Stokes (1819–1903) pointed out that the attenuation of surface waves could be exploited to measure viscosity. This paper provides the link between fluid viscosity and the attenuation of surface waves by invoking the conservation of energy. First we calculate the power loss per unit area due to viscous dissipation. Next we calculate the power loss per unit area as manifested in the decay of the wave amplitude. By equating these two quantities, we derive the relationship between the fluid viscosity and the decay coefficient of the surface waves in a transparent way.

1. Introduction

Surface tension and gravity govern the propagation of surface waves on fluids while viscosity determines the wave attenuation. In this paper we focus on the relation between viscosity and attenuation of surface waves.

More than a century ago, Stokes (1819–1903) pointed out that the attenuation of surface waves could be exploited to measure viscosity [1]. Since then, the determination of viscosity from the damping of surface waves has received much attention [2–10], particularly because the method presents the possibility of measuring viscosity noninvasively.

In his attempt to obtain the functional relationship between viscosity and wave attenuation, Stokes observed that the harmonic solutions obtained by solving the Laplace equation for the velocity potential in the absence of viscosity also satisfy the linearized Navier–Stokes equation [11]. However, to render the harmonic solutions suitable for viscous fluids, one must satisfy a new set of boundary conditions, which incorporate the viscous losses. The result is the introduction of a spatial decay in the wave amplitude.

To extract viscosity from the attenuation data, however, one needs to know the functional relationship between the decay coefficient of surface waves and the viscosity. Most authors resort to the result obtained by Stokes more than a century ago as quoted in Lamb [12], and more recently in Lighthill [11]. The derivations in these texts are based on modifying the harmonic solutions to satisfy the boundary conditions at the surface. What is more, most

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recent texts perpetuate this problem by simply quoting the old results without providing a modern derivation. Some texts simply avoid the subject [13].

This paper attempts to provide the link between the fluid viscosity and the decay coefficient of surface waves by a simple application of conservation of energy. First we calculate the power loss per unit area due to viscous dissipation. Next we calculate the power loss per unit area due to the decay of the wave amplitude. By equating these two quantities, we derive the relationship between the fluid viscosity and the decay coefficient of the surface waves in a transparent way.

In what follows we first give a brief outline of the potential theory and by a general argument arrive at the harmonic solutions of the Laplace equation and the associated surface waves for incompressible and irrotational flow. Next we modify these solutions to represent surface waves that decay in the presence of viscosity and derive the relation between viscosity and the decay coefficient of the waves.

2. Surface waves on incompressible and inviscid fluids

Starting with the differential form of the equation of continuity, where ρ is the fluid density and $\boldsymbol{v}$ is the velocity field, we have

$$\partial\rho/\partial t = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) \tag{1}$$

which, when expanded, takes the form

$$\partial\rho/\partial t + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = 0. \tag{2}$$

For an incompressible fluid ρ is constant, which in light of equation (2) implies that the velocity field is source free, i.e.,

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0. \tag{3}$$

Furthermore, for irrotational flow the velocity field is also curl free, i.e.,

$$\boldsymbol{\nabla}\times\boldsymbol{v} = 0. \tag{4}$$

Subject to the boundary conditions, equations (3) and (4) completely specify the vector field $\boldsymbol{v}$ to within a constant. Furthermore, equation (4) implies that the vector field $\boldsymbol{v}$ can be derived from a scalar potential ϕ through the relation,

$$\boldsymbol{v} = \boldsymbol{\nabla}\phi \tag{5}$$

which immediately leads to,

$$\boldsymbol{\nabla}^2\phi = 0. \tag{6}$$

Equation (6), first derived by Euler, is now known as Laplace's equation.

Laplace's equation is not a wave equation; however, it does admit of harmonic solutions with one caveat: it cannot describe the propagation of waves in an incompressible fluid if the system is either bounded by stationary surfaces or is infinite in extent. This is because a harmonic function of the form

$$\phi = \varphi_0 \mathrm{e}^{\mathrm{i}(\omega t \pm \boldsymbol{k}\cdot\boldsymbol{r})} \tag{7}$$

is a solution of the Laplace equation only if the wavevector $\boldsymbol{k}$ satisfies the relation $\boldsymbol{k}\cdot\boldsymbol{k} = 0$. Therefore, except for the trivial case of $\boldsymbol{k} = 0$, $\boldsymbol{k}$ must be complex. Here φ_0 is the potential amplitude, ω is the angular frequency, $\boldsymbol{r}$ is the position vector, and t stands for time.

However, when the system is bounded by fixed surfaces, a complex $\boldsymbol{k}$ is inadmissible because one cannot satisfy the boundary conditions. This is so because at a fixed boundary the velocity $\boldsymbol{v}$ must vanish, i.e.

$$\boldsymbol{v} = \boldsymbol{\nabla}\phi = \boldsymbol{k}\phi = 0$$

which returns us to the trivial $\boldsymbol{k} = 0$ solution. Physically, a finite system with fixed boundaries cannot support waves because the potential energy of the system is a constant.

For an infinite system with no boundaries, a complex wavevector presents another obstacle; it causes the velocity potential to grow without bound as $\boldsymbol{r}$ in equation (7) approaches infinity. But when the system is partly bounded by a free surface a complex $\boldsymbol{k}$ vector is admissible since the free surface allows a change in the potential energy of the system as the surface level moves up and down due to the wave motion.

To obtain the harmonic solutions resulting from a complex wavevector, it is convenient to choose a coordinate system in which the fluid surface forms the x–z-plane, and the y-axis is normal to the fluid surface. Without loss of generality we consider waves travelling along the x-axis. With this choice of coordinates, the harmonic potential of equation (7) satisfies the Laplace equation if we choose a complex wavevector of the form

$$\boldsymbol{k} = k_x\boldsymbol{i} + \mathrm{i}k_y\boldsymbol{j} \tag{8}$$

in which case,

$$\boldsymbol{k}\cdot\boldsymbol{k} = 0 = k_x^2 - k_y^2. \tag{9}$$

This choice of $\boldsymbol{k}$ has the requisite virtue in that k_x is real and thus it allows wave propagation in the x-direction, while the presence of the imaginary term, $\mathrm{i}k_y$, ensures that the harmonic solution satisfies Laplace's equation. In light of equation (9), we have

$$k_x = k_y = \pm k \tag{10}$$

where $k = 2\pi/\lambda$ is the (positive) wavenumber. Thus the harmonic solution takes the form

$$\phi = \varphi_0 \mathrm{e}^{\pm ky}\mathrm{e}^{\mathrm{i}(\omega t \pm kx)}. \tag{11}$$

However, since the y-coordinate is negative for points below the surface, only the e^{ky} solution guarantees a finite velocity potential as y increases with depth.

Consequently, the velocity potential representing a sinusoidal wave moving in the $+x$-direction may be written as

$$\phi = \varphi_0 \mathrm{e}^{ky}\cos(\omega t - kx). \tag{12}$$

Evidently, under wave action, the velocity of a fluid element whose equilibrium position is at (x, y) is given by

$$\boldsymbol{v} = \boldsymbol{\nabla}\phi = v_0\mathrm{e}^{ky}[\sin(\omega t - kx)\boldsymbol{i} + \cos(\omega t - kx)\boldsymbol{j}]. \tag{13}$$

Here $v_0 = k\varphi_0$ is the velocity amplitude at the surface.

Furthermore, the displacement $\boldsymbol{\psi}$ of a fluid element from its equilibrium position (x, y) is related to its velocity by

$$\partial\boldsymbol{\psi}/\partial t = \boldsymbol{v}. \tag{14}$$

Therefore, the displacement of a fluid element relative to its equilibrium position due to the wave motion is given by

$$\boldsymbol{\psi} = a\mathrm{e}^{ky}[-\cos(\omega t - kx)\boldsymbol{i} + \sin(\omega t - kx)\boldsymbol{j}] \tag{15}$$

where $a = v_0/\omega$ is the magnitude of the displacement of a water element at the free surface from its equilibrium. Note that a is also the wave amplitude at the surface. Furthermore, the displacement and velocity of a water element diminish exponentially with depth below the surface.

Equation (15) is a right-handed circular wave travelling in the positive x-direction with the wave speed of ω/k. Figure 1 is a representation of such a wave on the surface. In this schematic diagram the amplitude of the wave has been exaggerated for clarity. The solid arrows represent the velocity of a fluid element; the dashed arrows show the displacement of a fluid element from its equilibrium position. Each fluid element moves clockwise around a circular path of radius $a\mathrm{e}^{ky}$, centred on its equilibrium position.

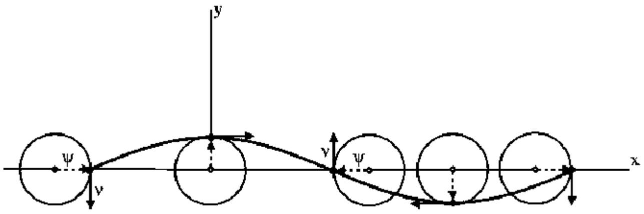

Figure 1. A schematic diagram of a right-handed circular wave travelling in the $+x$-direction. The dashed arrows show the displacement of the fluid elements from their equilibrium positions at the centre of each circle. The solid arrows represent the velocity of the displaced fluid elements. Each fluid element moves clockwise around a circular path of radius ae^{ky}, centred on its equilibrium position. The wave amplitude is vastly exaggerated for clarity.

3. Harmonic solutions and the equation of motion

We may well ask whether the harmonic solutions as expressed in equation (15) satisfy the equation of motion for a fluid element. To answer this question, let us consider the forces acting on a fluid element under wave action.

In the absence of viscosity, the net force on a closed volume element is due to pressure and gravity and is given by

$$\boldsymbol{f}_{\text{net}} = -\oint_S p\,\mathrm{d}\boldsymbol{S} + \int \rho\boldsymbol{g}\,\mathrm{d}V = \int(-\boldsymbol{\nabla}p + \rho\boldsymbol{g})\,\mathrm{d}V. \tag{16}$$

Here p is the pressure, and $\boldsymbol{g}$ is the acceleration of gravity. Evidently, the net force per unit volume is given by the expression $(-\boldsymbol{\nabla}p + \rho\boldsymbol{g})$. Under hydrostatic equilibrium, the net force per unit volume is zero, i.e., $\boldsymbol{\nabla}p = \rho\boldsymbol{g}$. But under wave action, the pressure experienced by a fluid element within the liquid departs from its hydrostatic value by an excess pressure p_e. This excess pressure is due to the vertical displacement of the fluid element from its equilibrium position. In other words, $p_\mathrm{e} = \rho g\psi_j$, where g is the magnitude of the acceleration of gravity and ψ_j is the vertical component of the displacement of a fluid element from its equilibrium position due to the wave motion.

Under the wave action, equation (15) gives the displacement of a fluid element from its equilibrium position, where the vertical component of this displacement is

$$\psi_j = a\mathrm{e}^{ky}\sin(\omega t - kx).$$

Therefore, for a liquid element whose equilibrium position is located at the point (x, y), the excess pressure p_e is given by

$$p_\mathrm{e} = \rho g\psi_j = \rho ga\mathrm{e}^{ky}\sin(\omega t - kx). \tag{17}$$

However, by equation (16) the net force per unit volume due to this excess pressure is $-\boldsymbol{\nabla}p_\mathrm{e}$. Therefore, the equation of motion for a fluid element is simply given by

$$\rho\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = \rho[\partial\boldsymbol{v}/\partial t + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}] = -\boldsymbol{\nabla}p_\mathrm{e}. \tag{18}$$

When the amplitude of the wave is small relative to the wavelength, the nonlinear term in equation (18) may be dropped to obtain the linearized Euler equation.

$$\rho[(\partial\boldsymbol{v}/\partial t)] = -\boldsymbol{\nabla}p_\mathrm{e}. \tag{19}$$

Does the harmonic wave solution satisfy the linearized Euler equation? Substitution of equations (13) and (17) in (19) gives

$$\rho[(\partial\boldsymbol{v}/\partial t)] = \rho\omega v_0\mathrm{e}^{ky}[\cos(\omega t - kx)\boldsymbol{i} - \sin(\omega t - kx)\boldsymbol{j}]$$
$$-\boldsymbol{\nabla}p_\mathrm{e} = \rho gak[\mathrm{e}^{ky}[\cos(\omega t - kx)\boldsymbol{i} - \sin(\omega t - kx)\boldsymbol{j}]].$$

Since $v_0 = a\omega$, the two expressions are equal if

$$\omega^2 = kg. \quad (20)$$

This is indeed the dispersion relation for gravity waves on deep water [14]. Thus the harmonic wave solution as given by equation (15) satisfies the linearized equation of motion of a fluid element.

To take account of the viscous drag forces, equation (18) must be modified. The net viscous force per unit volume is given by the expression $\eta\nabla^2\boldsymbol{v}$, where η stands for viscosity [15]. Equation (18) may thus be generalized to

$$\rho[(\partial\boldsymbol{v}/\partial t) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] = -\nabla p_\mathrm{e} + \eta\nabla^2\boldsymbol{v}. \quad (21)$$

This is, of course, the Navier–Stokes equation for incompressible flow. When the amplitude of the wave is small relative to the wavelength, the nonlinear term in equation (21) may be dropped to yield the linearized version of the Navier–Stokes equation.

$$\rho(\partial\boldsymbol{v}/\partial t) = -\nabla p_\mathrm{e} + \eta\nabla^2\boldsymbol{v}. \quad (22)$$

Does the harmonic wave solution given in equation (15) satisfy the linearized Navier–Stokes equation? Stokes was first to observe [11] that the viscous term, $\eta\nabla^2\boldsymbol{v}$, vanishes identically when equation (13) represents the velocity field $\boldsymbol{v}$, a fact that can be verified easily by using $\boldsymbol{v}$ as given in equation (13) to evaluate the expression $\nabla^2\boldsymbol{v}$ [16].

4. Viscosity and wave attenuation

Equations (13) and (15) represent irrotational wave motion in an incompressible fluid when the fluid depth is larger than the wavelength λ. They are the harmonic solutions of Laplace's equation for irrotational flow and, as discussed above, do satisfy the linear version of the equation of motion for a fluid element,

Furthermore, even in the presence of viscosity these same harmonic solutions satisfy the linearized Navier–Stokes equation. However, in this case the amplitude of the wave decays due to viscous losses. The energy dissipation, manifested in the amplitude decay, is due to viscous forces. Stokes was first to recognize that this energy loss is numerically equal to the work of an external force which, when applied to the free surface of the fluid, would counteract the viscous drag forces and assure unattenuated irrotational motion [11]. One can appreciate Stokes' insight by observing wind driven waves on lakes.

The external applied force must balance only two components of the stress tensor σ_{ij}. The shear stress is given by σ_{xy}, which points in the x-direction, while the normal stress is given by σ_{yy}, which points in the y-direction. Note that there is no shear stress in the x–z-plane along the z-direction as there is no z-dependence in the equations of motion. More explicitly,

$$\sigma_{xy} = -\eta[(\partial v_y/\partial x) + (\partial v_x/\partial y)], \quad (23)$$

and

$$\sigma_{yy} = p_\mathrm{e} - 2\eta[(\partial v_y/\partial y)]. \quad (24)$$

When the expression for $\boldsymbol{v}$ as given in equation (13) is used in equations (23) and (24), the results are

$$\boldsymbol{\sigma_{xy}} = -2\eta a\omega k\mathrm{e}^{ky}\cos(\omega t - kx)\boldsymbol{i} \quad (25)$$

and

$$\boldsymbol{\sigma_{yy}} = [p_\mathrm{e} + 2\eta a\omega k\mathrm{e}^{ky}\sin(\omega t - kx)]\boldsymbol{j}. \quad (26)$$

Consequently, the net power dissipated per unit surface area as the wave moves on is simply given by the vector product of these two stress components and the fluid velocity at the surface. This may be written most conveniently as

$$\mathrm{d}P/\mathrm{d}A = (\sigma_{xy} + \sigma_{yy})\cdot\boldsymbol{v}. \quad (27)$$

When equations (13), (17), (25), and (26) are substituted into equation (27), one obtains

$$\mathrm{d}P/\mathrm{d}A = -2\eta a^2\omega^2 k \mathrm{e}^{2ky}\cos^2(\omega t - kx) + [-a\omega\rho g a\cos(\omega t - kx)\sin(\omega t - kx) - 2\eta a^2\omega^2 k \mathrm{e}^{2ky}\sin^2(\omega t - kx)]$$

which simplifies to the following expression for the average power dissipation per unit area at the surface:

$$\langle \mathrm{d}P/\mathrm{d}A\rangle = -2\eta a^2\omega^2 k. \tag{28}$$

Note that equation (28) implies that when viscosity is negligible, the power loss per unit area is zero as expected. On the other hand, in the presence of viscosity, the wave amplitude decays exponentially with distance according to $a = a_0\mathrm{e}^{-\alpha x}$, where a_0 is the wave amplitude at $x = 0$ and α is the decay coefficient. Therefore the power loss per unit area is a function of position. More explicitly,

$$\langle \mathrm{d}P/\mathrm{d}A\rangle = -2\eta a_0^2\mathrm{e}^{-2\alpha x}\omega^2 k. \tag{29}$$

Consequently, since the wave energy diminishes as the wave moves along, the wave energy per unit area E is also a function of position.

Furthermore, we note that in a travelling wave the kinetic energy represents only half the total energy of the wave since the wave energy is equally divided between kinetic and potential forms. This equipartition of energy between the kinetic and potential forms is a general property of harmonic waves in material media. For water waves the equipartition is most easily demonstrated by considering the case of a standing wave formed by the superposition of two travelling waves of the same amplitude and frequency moving in the opposite directions. Here the entire energy of the standing wave is kinetic when the wave is in the state where the fluid surface is flat, and the energy is entirely potential when the wave attains its maximum amplitude. Consequently, in a standing wave the average kinetic and potential energies are equal. Note, however, that whereas in a standing wave the energy shifts between kinetic and potential forms, in a travelling wave the energy is divided equally between the kinetic and potential forms. We mention in passing that in a previous article we have exploited the transformation of energy from kinetic to potential forms in standing waves to derive the general dispersion relation of water waves based on conservation of energy [17].

Therefore, since the velocity amplitude at a point (x, y) is given by $v = a\omega\mathrm{e}^{-\alpha x}\mathrm{e}^{ky}$, we conclude

$$E = 2\int_0^{-\infty}(1/2\rho v^2)\,\mathrm{d}y = \int_0^{-\infty}\rho a_0^2\omega^2\mathrm{e}^{-2\alpha x}\mathrm{e}^{2ky}\,\mathrm{d}y = \rho a_0^2\omega^2\mathrm{e}^{-2\alpha x}/2k. \tag{30}$$

Consequently,

$$\mathrm{d}E/\mathrm{d}x = -2\alpha E \tag{31}$$

and

$$\mathrm{d}E/\mathrm{d}t = -2\alpha E\,\mathrm{d}x/\mathrm{d}t = -2\alpha E v_{\mathrm{g}} = -2\alpha(\rho a_0^2\omega^2\mathrm{e}^{-2\alpha x}/2k)v_{\mathrm{g}} \tag{32}$$

where v_{g} is the group velocity.

Comparing equation (32) with the earlier expression for viscous power loss per unit area (equation (29)) we conclude that

$$-2\alpha(\rho a_0^2\omega^2\mathrm{e}^{-2\alpha x}/2k)v_{\mathrm{g}} = -2\eta a_0^2\mathrm{e}^{-2\alpha x}\omega^2 k$$

and thus

$$\eta = \rho v_{\mathrm{g}}\alpha/2k^2. \tag{33}$$

5. Discussion

In deriving equation (33), we have simply invoked the conservation of energy. First we obtained the power loss per unit area due to viscous stress components in the surface layer. Next we calculated the power loss of the wave per unit area due to its amplitude attenuation. By equating these two losses, we derived the relation between viscosity and decay coefficient.

Equation (33) provides a means for determining the viscosity of a fluid from a measurement of the decay coefficient of surface waves. However, to complete the task one must also have the group velocity of the surface waves. The group velocity is defined by the expression

$$v_g = d\omega/dk.$$

The simple dispersion relation given in equation (20) is only valid for gravity waves where the effect of surface tension may be neglected. The more general dispersion relation for surface waves which takes the effect of surface tension into account is given by the expression

$$\omega^2 = kg + \sigma k^3/\rho \tag{34}$$

where σ is the surface tension. The group velocity is therefore given by differentiating equation (34) to get

$$v_g = (g + 3\sigma k^2/\rho)/2(kg + \sigma k^3/\rho)^{1/2}. \tag{35}$$

In practice, equation (33) takes a simpler form depending on the wave regime under study. For capillary waves the second term on the right-hand side of equation (34) dominates $[k^2/g \gg 1]$, and the dispersion relation takes the form

$$\omega^2 \cong \sigma k^3/\rho$$

and the group velocity $v_g = (3\omega/2k)$. So equation (33) takes the form

$$\eta = 3\rho\omega\alpha/4k^3 \qquad \text{(capillary waves)}.$$

For gravity waves, the first term on the right-hand side of equation (34) dominates, resulting in $\omega^2 \cong kg$ as the dispersion relation for gravity waves, which in turn results in $v_g = (\omega/2k)$. In this case equation (33) takes the form

$$\eta = \rho\omega\alpha/4k^3 \qquad \text{(gravity waves)}.$$

In the intermediate wave regime known as capillary-gravity waves, both gravity and surface tension terms in the dispersion relation must be taken into account. When equation (35) is used in equation (33), we obtain the viscosity in terms of four measurable quantities, namely wavenumber $k = 2\pi/\lambda$, surface tension σ, density ρ, and the attenuation coefficient α. Indeed, we have

$$\eta = [\alpha\rho/2k^2][(g + 3\sigma k^2/\rho)/2(kg + \sigma k^3/\rho)^{1/2}]. \tag{36}$$

Experimentally, the attenuation coefficient α is obtained from a plot of the wave amplitude versus distance travelled. Reference [10] describes how equation (36) is used in practice to obtain the viscosity of water as a function of temperature by measuring the decay coefficient of capillary-gravity waves.

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