Chapter 7 Problems

E3200x Mechanics

7-4. A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α (> 0) are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

7-6. A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Find the Lagrange equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

7-8. Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 it is U_2. If a particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2 such that its path in region 1 makes an angle θ_1 with the normal to the plane of separation and an angle θ_2 with the normal when in region 2, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \left(1 + \frac{U_1 - U_2}{T_1}\right)^{1/2}$$

where $T_1 = \frac{1}{2}mv_1^2$. What is the optical analog of this problem?

7-10. Two blocks, each of mass M, are connected by an extensionless, uniform string of length l. One block is placed on a smooth horizontal surface, and the other block hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system **(a)** when the mass of the string is negligible and **(b)** when the string has a mass m.

7-12. A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

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4. $m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} = 0;\quad \frac{d}{dt}(mr^2\dot{\theta}) = 0;$ yes; yes

6. $2m\ddot{S} + m\ddot{\xi}\cos\alpha - mg\sin\alpha = 0$

$(m + M)\ddot{\xi} + m\ddot{S}\cos\alpha = 0$

10. (a) $y(t) = -\frac{g}{4}t^2$ (b) $y(t) = \frac{Ml}{m}(1 - \cosh\gamma t)$

12. $r(t) = r_0\cosh\alpha t + \frac{g}{2\alpha^2}(\sin\alpha t - \sinh\alpha t)$

7-14. A simple pendulum of length b and bob with mass m is attached to a massless support moving vertically upward with constant acceleration a. Determine (a) the equations of motion and (b) the period for small oscillations.

7-16. The point of support of a simple pendulum of mass m and length b is driven horizontally by $x = a \sin \omega t$. Find the pendulum's equation of motion.

7-18. A pendulum is constructed by attaching a mass m to an extensionless string of length l. The upper end of the string is connected to the uppermost point on a vertical disk of radius R ($R < l/\pi$) as in Figure 7-B. Obtain the pendulum's equation of motion, and find the frequency of small oscillations. Find the line about which the angular motion extends equally in either direction (i.e., $\theta_1 = \theta_2$).

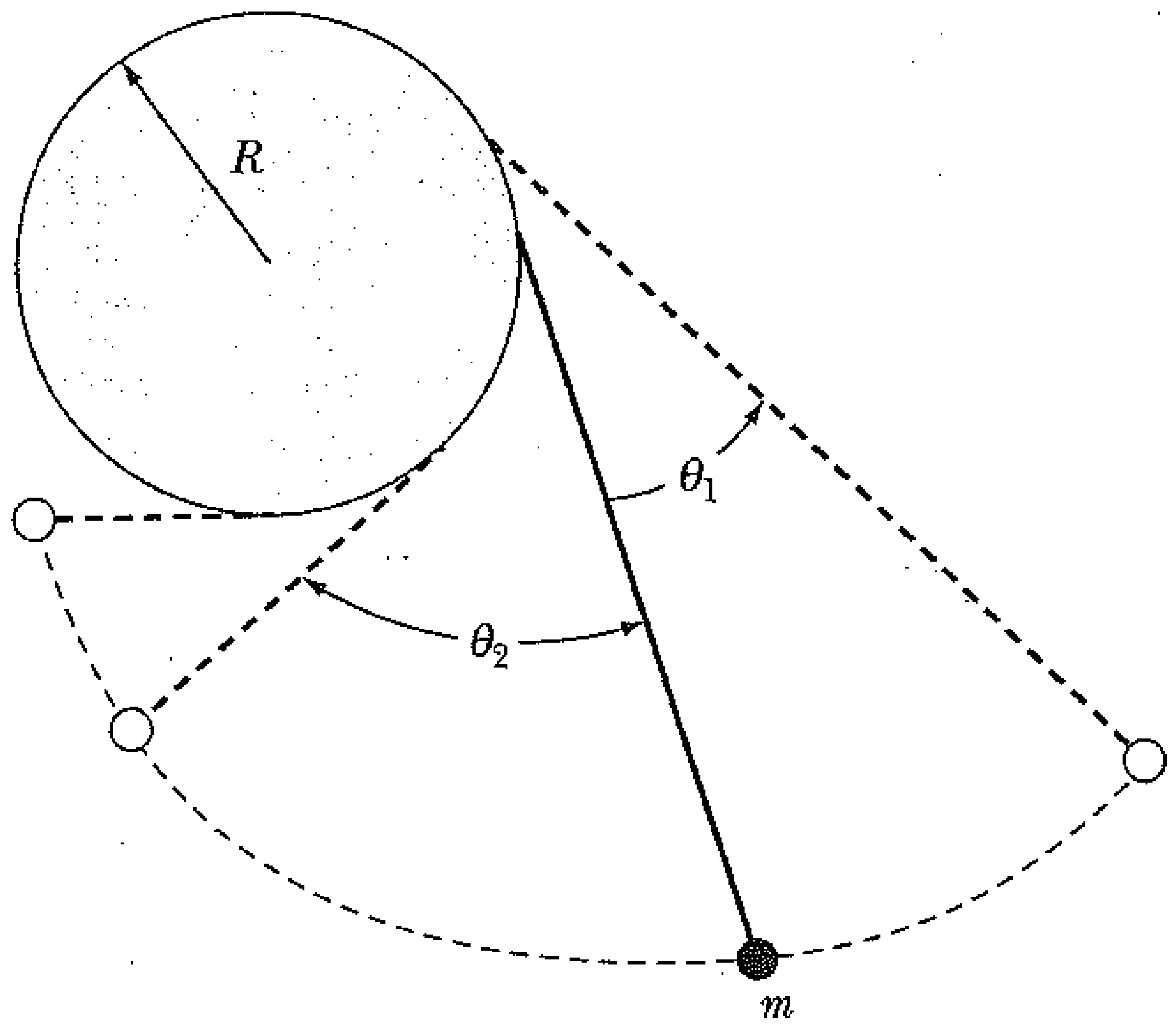

FIGURE 7-B Problem 7-18.

14. (a) $\ddot{\theta} + \dfrac{a+g}{b} \sin \theta = 0$ (b) $2\pi \sqrt{\dfrac{b}{a+g}}$

16. $\ddot{\theta} + \dfrac{g}{b} \sin \theta - \dfrac{a}{b} \omega^2 \sin \omega t \cos \theta = 0$

18. $\omega = \sqrt{\dfrac{g \sin \theta_0}{l - R\theta_0}}; \quad \theta_0 = \dfrac{\pi}{2}$

22. $L = \dfrac{1}{2} m\dot{x}^2 - \dfrac{k}{x} e^{-t/\tau}; \quad H = \dfrac{p_x^2}{2m} + \dfrac{k}{x} e^{-t/\tau}$

7-22. A particle of mass m moves in one dimension under the influence of a force

$$F(x, t) = \frac{k}{x^2} e^{-(t/\tau)}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

7-24. Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

7-26. Determine the Hamiltonian and Hamilton's equations of motion for **(a)** a simple pendulum and **(b)** a simple Atwood machine (single pulley).

7-28. A particle of mass m is attracted to a force center with the force of magnitude k/r^2. Use plane polar coordinates and find Hamilton's equations of motion.

7-32. A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface $H =$ constant in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four dimensional (r, θ, p_r, p_θ, but only the first three are nontrivial). Calculate the projection of the phase path on the r-p_r plane, then take into account the variation with θ.

24. $L = \frac{1}{2}m(\alpha^2 + l^2\dot{\theta}^2) + mgl\cos\theta$

$$H = \frac{p_\theta^2}{2ml^2} - \frac{1}{2}m\alpha^2 - mgl\cos\theta$$

26. (a) $H = \dfrac{p_\theta^2}{2ml^2} - mgl\cos\theta;\quad \dot{\theta} = \dfrac{p_\theta}{ml^2};\ \dot{p}_\theta = -mgl\sin\theta$

(b) $H = \dfrac{p_x^2}{2(m_1 + m_2 + I/a^2)} - m_1gx - m_2g(l - x)$

$$p_x = \left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}$$

$$\dot{p}_x = g(m_1 - m_2)$$

28. $p_r = m\dot{r};\ \dot{p}_r = \dfrac{p_\theta^2}{mr^3} - \dfrac{k}{r^2};\ p_\theta = mr^2\dot{\theta};\ \dot{p}_\theta = 0$

32. $H = \dfrac{1}{2m}\left(p_r^2 + \dfrac{p_\theta^2}{r^2} + \dfrac{p_\phi^2}{r^2\sin^2\theta}\right) - \dfrac{k}{r};\quad \dot{p}_r = -\dfrac{k}{r^2} + \dfrac{p_\theta^2}{mr^3} + \dfrac{p_\phi^2}{mr^3\sin^2\theta};$

$$\dot{p}_\theta = \frac{p_\phi^2\cot\theta}{mr^2\sin^2\theta};\ \dot{p}_\phi = 0$$

7-32. A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface $H =$ constant in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four dimensional (r, θ, p_r, p_θ, but only the first three are nontrivial). Calculate the projection of the phase path on the r-p_r plane, then take into account the variation with θ.

7-34. A particle of mass m slides down a smooth circular wedge of mass M as shown in Figure 7-C. The wedge rests on a smooth horizontal table. Find (a) the equation of motion of m and M and (b) the reaction of the wedge on m.

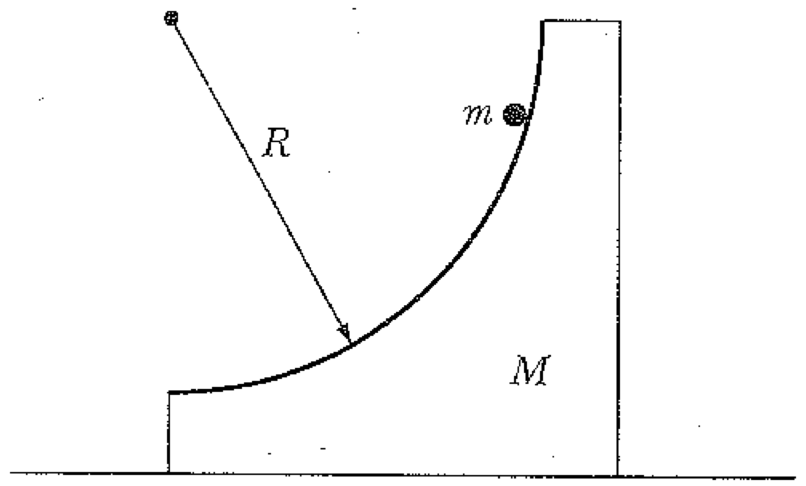

FIGURE 7-C Problem 7-34.

7-38. The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where k and b are constants. Find Hamilton's equations of motion.

7-40. A double pendulum is attached to a cart of mass $2m$ that moves without friction on a horizontal surface. See Figure 7-D. Each pendulum has length b and mass bob m. Find the equations of motion.

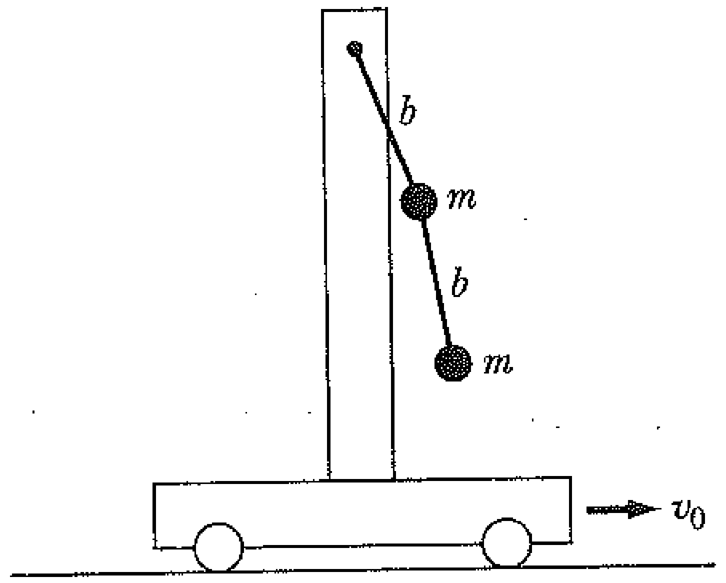

FIGURE 7-D Problem 7-40.

32. $H = \dfrac{1}{2m}\left(p_r^2 + \dfrac{p_\theta^2}{r^2} + \dfrac{p_\phi^2}{r^2 \sin^2\theta}\right) - \dfrac{k}{r}$; $\dot{p}_r = -\dfrac{k}{r^2} + \dfrac{p_\theta^2}{mr^3} + \dfrac{p_\phi^2}{mr^3 \sin^2\theta}$;

$\dot{p}_\theta = \dfrac{p_\phi^2 \cot\theta}{mr^2 \sin^2\theta}$; $\dot{p}_\phi = 0$

34. (a) $\ddot{x} = aR(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$; $\ddot{\theta} = \dfrac{\ddot{x}\sin\theta + g\cos\theta}{R}$; where $a \equiv \dfrac{m}{M+m}$

(b) $\lambda = -\dfrac{mMg(3\sin\theta - a\sin^3\theta - 2\sin\theta_0)}{(M+m)(1 - a\sin^2\theta)^2}$

38. $\dfrac{dx}{dt} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m}$, $\dfrac{dp}{dt} = -\dfrac{\partial H}{\partial x} = -(kx + bx^3)$

40. $0 = 4\dfrac{d^2x}{dt^2} + b\left(2\dfrac{d^2\theta_1}{dt^2}\cos\theta_1 + \dfrac{d^2\theta_2}{dt^2}\cos\theta_2\right) - b\left(2\left(\dfrac{d\theta_1}{dt}\right)^2 \sin\theta_1 + \left(\dfrac{d\theta_2}{dt}\right)^2 \sin\theta_2\right)$

$$-2g\sin\theta_1 = 2b\frac{d^2\theta_1}{dt^2} + 2\frac{d^2x}{dt^2}\cos\theta_1 + b\frac{d^2\theta_2}{dt^2}\cos(\theta_1 - \theta_2) + b\left(\frac{d\theta_2}{dt}\right)^2 \sin(\theta_1 - \theta_2)$$

$$-g\sin\theta_2 = b\frac{d^2\theta_2}{dt^2} + \frac{d^2x}{dt^2}\cos\theta_2 + b\frac{d^2\theta_1}{dt^2}\cos(\theta_1 - \theta_2) - b\left(\frac{d\theta_1}{dt}\right)^2 \sin(\theta_1 - \theta_2)$$