Moon-Earth-Sun: The oldest three-body problem

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The daily motion of the Moon through the sky has many unusual features that a careful observer can discover without the help of instruments. The three different frequencies for the three degrees of freedom have been known very accurately for 3000 years, and the geometric explanation of the Greek astronomers was basically correct. Whereas Kepler's laws are sufficient for describing the motion of the planets around the Sun, even the most obvious facts about the lunar motion cannot be understood without the gravitational attraction of both the Earth and the Sun. Newton discussed this problem at great length, and with mixed success; it was the only testing ground for his Universal Gravitation. This background for today's many-body theory is discussed in some detail because all the guiding principles for our understanding can be traced to the earliest developments of astronomy. They are the oldest results of scientific inquiry, and they were the first ones to be confirmed by the great physicist-mathematicians of the 18th century. By a variety of methods, Laplace was able to claim complete agreement of celestial mechanics with the astronomical observations. Lagrange initiated a new trend wherein the mathematical problems of mechanics could all be solved by the same uniform process; canonical transformations eventually won the field. They were used for the first time on a large scale by Delaunay to find the ultimate solution of the lunar problem by perturbing the solution of the two-body Earth-Moon problem. Hill then treated the lunar trajectory as a displacement from a periodic orbit that is an exact solution of a restricted three-body problem. Newton's difficultly in explaining the motion of the lunar perigee was finally resolved, and the Moon's orbit was computed by a new method that became the universal standard until after WW II. Poincaré opened the 20th century with his analysis of trajectories in phase space, his insistence on investigating periodic orbits even in ergodic systems, and his critique of perturbation theory, particularly in the case of the Moon's motion. Space exploration, astrophysics, and the landing of the astronauts on the Moon led to a new flowering of celestial mechanics. Lunar theory now has to confront many new data beyond the simple three-body problem in order to improve its accuracy below the precision of 1 arcsecond; the computer dominates all the theoretical advances. This review is intended as a case study of the many stages that characterize the slow development of a problem in physics from simple observations through many forms of explanation to a high-precision fit with the data. [S0034-6861(98)00802-2]

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I. INTRODUCTION

If there be nothing new, but that which is
Hath been before, how are our brains beguiled,
Which, laboring for invention, bear amiss
The second burden of a former child!
(Shakespeare, Sonnet 59)

A. The Moon as the first object of pure science

When we try to understand a special area in physics ourselves, or when we teach the basics of some specialty to our students, there is no better way than to go through the most important steps in their historical order. While doing so, it would be a pity if we did not make comparisons with the historic progression in related fields and identify the common features that help us to establish a successful and convincing picture in any area. The Moon's motion around the Earth offers us the prime example in this respect.

Although we think primarily of the planets orbiting the Sun as the fundamental issue for the origin of modern science, it was really the Moon that provided the principal ideas as well as the crucial tests for our understanding of the universe. For simplicity's sake I shall distinguish three stages in the development of any particular scientific endeavor. In all three of them the Moon played the role of the indispensable guide without whom we might not have found our way through the maze of possibilities.

The first stage of any scientific achievement was reached 3000 years ago in Mesopotamia when elementary observations of the Moon on the horizon were made and recorded. The relevant numbers were then represented by simple arithmetic formulas that lack any insight in terms of geometric models, let alone physical principles. And yet, most physicists are not aware of the important characteristic frequencies in the lunar orbit that were discovered at that time. They can be compared with the masses of elementary particles, our present-day understanding of which hardly goes beyond their numerical values.

The second stage was initiated by the early Greek philosophers, who thought of the universe as a large empty space with the Earth floating at its center, the Sun, the Moon, and the planets moving in their various orbits around the center in front of the background of fixed stars. This grand view may have been the single most significant achievement of the human mind. Without the Moon, visible both during the night and during the day, it is hard to imagine how the Sun could have been conceived as moving through the Zodiac just like the Moon and the planets. The Greek mathematicians and astronomers were eventually led to sophisticated geometric models that gave exact descriptions without any hint of the underlying physics.

The third stage was reached toward the end of the 17th century with the work of Isaac Newton. His grand opus, *The Mathematical Principles of Natural Philosophy*, represents the first endeavor to explain observations both on Earth and in the heavens on the basis of a few physical "laws" in the form of mathematical relations. The crucial test is the motion of the Moon together with several related phenomena such as the tides and the precession of the equinoxes. This first effort at unification can be called a success only because it was able to solve some difficult problems such as the interaction of the three bodies Moon-Earth-Sun.

Modern physics undoubtedly claims to have passed already through stages one and two, but have we reached stage three in such areas as elementary-particle physics or cosmology? Can we match Newton's feat of finding two mathematical relations between the four relevant lunar and solar frequencies that were known in antiquity to five significant decimals? The gravitational three-body problem has provided the testing ground for many new approaches in the three centuries since Newton. But we are left with the question: what are we looking for in our pursuit of physics?

The Moon as well as elementary particles and cosmology are problems whose solutions can be called quite remote and useless in today's world. That very quality of detachment from everyday life makes them prime examples of pure science, objects of curiosity without apparent purpose, such as only human beings would find interesting. In following the development in the case of the Moon over the past three centuries we get an idea of what is in store for us in other fields.

B. Plan of this review

The three principal coordinate systems in the sky are described in Sec. II; they are based on the local horizon, on the equator, and on the ecliptic. The relations between these coordinate systems are fundamental for understanding the process of observing and interpreting the results of the observations. The various definitions of time in astronomy are recalled, as well as the measurement of linear distances in the solar system, which plays a special role in celestial mechanics because the equations of motion scale with the distance.

Section III is devoted to the prescientific and the earliest scientific achievements in the search for understanding of the lunar motions. The more obvious features that are easily seen with the bare eye can then be explained. The basic periods can be obtained from observations near the horizon, i.e., near the time of moonrise or moonset. The Babylonians in the last millennium B.C. developed a purely numerical scheme for predicting the important events in the lunar cycle. Their great achievement was the precision measurement of the various fundamental frequencies in the Moon's motion.

In the last few centuries B.C., the Greeks developed a picture of the universe that is still essentially valid in our time. The solar system is imbedded in a large three-dimensional vacuum, which is surrounded by the fixed stars. The Sun, the Moon, and the five classical planets move along rather elaborate orbits of various sizes, the Moon being by far the closest to the Earth. The basic physics such as the conservation of angular momentum is hidden in these purely geometric models. The discussion in Sec. IV will hardly do justice to this great advance in our understanding of the universe around us.

Some of the Greek data were refined by Islamic astronomers, but even the awakening in the 15th and 16th century, in particular the great treatise by Copernicus, did not improve the calculation of the Moon's motion, as will be shown in Sec. V. Physics came into the picture when Kepler got a chance to interpret Brahe's data, and Galileo had the marvelous idea of looking at the stars with the newly invented telescope. The explosive accumulation of observations without a useful theory led to stagnation at the end of the 17th century, not unlike high-energy physics at the end of the 20th century.

The big breakthrough came with the publication of the *Principia* by Newton in 1687. The theory of the Moon became the great test for Newton's laws of motion and universal gravitation, as will be described in Sec. VI. His truly awesome (and generally quite unappreciated) results in this area are well worth explaining in detail before getting into the inevitable technical refinements that are needed to exploit fully what Newton had only tentatively suggested.

The great mathematicians of the 18th century succeeded in clearing up Newton's difficulties with calculating the motion of the lunar perigee. Section VII tries to give an idea of their straightforward, but somewhat clumsy, methods. Laplace was able to carry out all the necessary calculations, but his grand unification of all celestial mechanics came at a high price; physics was again in danger of getting lost.

The next three sections are more technical in content. They try to provide a glimpse of the general methods that were proposed in order to deal with the difficult three-body problem Moon-Earth-Sun. Lagrange's idea of "varying the constants" is discussed in Sec. VIII; a few successful examples of his method still retain some intuitive appeal. Section IX describes the origin of the canonical formalism and its first use on a grand scale by Delaunay to find the lunar trajectory. Poincaré showed the ultimate futility of any expansion in the critical parameter that caused Newton so much grief. Toward the end of the 19th century Hill approached lunar theory by starting with a periodic orbit of a perturbed dynamical system. The many advantages of this method are discussed in Sec. X, including the complete ephemeris of Brown and Eckert that was basic for the Apollo program and the implementation on a modern computer.

A somewhat haphazard survey of other developments in the 20th century, as well as some older but timely problems related to the Moon, are brought up in Sec. XI. Modern technology in connection with the space program is responsible for many improvements both in observing and in understanding the dynamic system Moon-Earth-Sun. High-speed computers have moved the emphasis away from the general theory of the three-body problem toward a better look at the detailed features in many of its special cases. We have come full-circle back to watching elementary phenomena, but they present themselves on the screen of a monitor rather than as the Sun or the Moon on the local horizon.

II. COORDINATES IN THE SKY

A. The geometry of the solar system

Any account of the motions in the Moon-Earth-Sun system has to start by defining the basic coordinates. Our cursory discussion describes the technical aspect of the Greek universe and still represents the fundamental approach to running a modern observatory. The geocentric viewpoint is unavoidable as long as we are dependent on telescopes that are fixed on the ground or are attached to a satellite.

For more details the reader is encouraged to take a look at the *Explanatory Supplement to the Astronomical Ephemeris*, a 500-page volume that is published by the US Naval Observatory and the Royal Greenwich Observatory. Among the many introductory texts on spherical astronomy are the classics by Smart (1931) and Woolard and Clemence (1966). The reader may also find some useful explanations in more elementary textbooks like those of Motz and Duveen (1977) and Roy (1978).

Three conceivable places for the observer can serve as the origin of a coordinate-system: (i) topocentric, from the place of the observatory on the surface of the Earth, (ii) geocentric, from the center of the Earth, or (iii) heliocentric, from the center of the solar system. Each coordinate system on the celestial sphere requires for its definition a plane or, equivalently, a direction perpendicular to the plane. A full polar coordinate system is obtained by adding the distance from the observer.

The whole machinery of these reference systems was the invention of the Greeks as part of their purely geometric view of the universe; it is a crucial preliminary step toward our physical picture of the world. The construction is completed by defining a full-fledged Cartesian coordinate system, which is centered on one of the three possible locations for the observer. The ordinary formulas for the transformations from one Cartesian system to another can be used, rather than the less familiar formulas from spherical trigonometry.

B. Azimuth and altitude—declination and hour angle

The local plumbline fixes the point Z overhead, the (local) *zenith*, on the imaginary spherical surface around us. The horizon is composed of all the points with a zenith distance equal to 90°. The center P for the apparent daily rotation is the north celestial pole; its distance from the zenith is the *colatitude* $\bar{\phi}=90°-\phi$ where ϕ is the geographical latitude for the place of observation. A great circle through P and Z defines the *north point N* of the horizon, as well as the other three *cardinal points* on the horizon, *east* (E), *south* (S), *west* (W).

The location of an object X is defined by moving west from the south point on the horizon through the *azimuth* ψ, and then straight up through the *altitude* ζ, counted positive toward the zenith. The altitude ζ of an object determines its visibility at the location of the observer. More importantly, the refraction of the light rays in the Earth's atmosphere is responsible for increasing the observed value of the altitude by as much as 30′ when the object is near the horizon. The refraction was not understood by the Greeks; nor even by Tycho Brahe, who gives different values for the Sun and for the Moon, although the values themselves are quite good.

The second spherical coordinate system has the main direction pointing toward the north celestial pole P. Any half of a great circle from the north pole to the south pole is called a *meridian*. The directions at right angle to the north pole form the *celestial equator*, which intersects the horizon in the two cardinal points east and west. Each star X has its own meridian. The *declination* δ is the angular distance of X from the equator, as measured along its meridian, positive to the north and negative to the south of the equator. The meridian through the zenith Z and S is the *observer's meridian*.

When the meridian of X coincides with the observer's meridian, the star is said to *transit* or *culminate*. At that moment the altitude of X is greatest, and the correction for refraction is least. The star's position is defined by the angle between its meridian and the observer's meridian, called the *hour angle* χ; it is measured from the observer's meridian toward west. The hour angle of any star increases from 0 at its transit to exactly $360°=24$ hours at its next transit, after one *sidereal day*.

The Sun goes around the celestial sphere once in one year, moving in a direction opposite to the daily motion of the fixed stars. It makes on the average only 365.25 transits in one year, whereas every fixed star makes 366.25 transits. Therefore, the sidereal day is shorter by 1/366.25, or a little less than 4 minutes, than the mean solar day.

The horizon system with the x,y,z axes pointing south, east, and toward the zenith, can be transformed into the equatorial system with the x',y',z' axes pointing south on the equator, east on the horizon, and toward the north pole, by a rotation around the common y axis through the colatitude $\bar{\phi}=90°-\phi$. The position of a star in the horizon system is given by the coordinates $(x,y,z)=(\cos\zeta\cos\psi,-\cos\zeta\sin\psi,\sin\zeta)$, whereas in the equatorial system $(x',y',z')=(\cos\delta\cos\chi, -\cos\delta\sin\chi,\sin\delta)$. The transformation is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\bar{\phi} & 0 & +\sin\bar{\phi} \\ 0 & 1 & 0 \\ -\sin\bar{\phi} & 0 & \cos\bar{\phi} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1}$$

A star is said to *rise* when its altitude becomes positive by passing the eastern half of the horizon, and to *set* as it dips into the western half of the horizon. The azimuth ψ_0 of a star with declination δ at the time of its setting, i.e., when its altitude $\zeta=0$, follows from

$$\sin(\psi_0-90°)=\frac{\sin(\delta)}{\cos(\phi)}. \tag{2}$$

At the latitude of New York City $\phi=40°$, and for $\delta=23.5°$ which is the declination of the Sun at the summer solstice, we find that the Sun sets $\psi_0-90°=31°22'$ on the horizon toward north from the point west.

The upper limit for the declination of the Moon has a cycle of about 18 years, during which it varies from $18°10'$ to $28°50'$. The corresponding moonsets, therefore, which are easily observed with the naked eye, vary from 24° north from the point west all the way to 39° at the latitude of New York.

C. Right ascension—longitude and latitude

The difference in hour angles between two fixed stars remains constant. Any feature Q on the celestial sphere may serve as a reference; its hour angle is called the *sidereal time* τ. The *right ascension* α of any star X is then defined by

$$\alpha=\tau-\chi. \tag{3}$$

The right ascension α increases toward the East as measured from Q, opposite the apparent motion of the celestial sphere.

The apparent path of the Sun through the sky with respect to the fixed stars is a great circle, which is called the *ecliptic*, with the pole K in the northern hemisphere; it changes only very slowly with time. An object X in the sky can be found by starting from the reference point Q and going east along the ecliptic through the *longitude* λ, and then toward K through the *latitude* β.

This reference point Q is generally chosen to be the vernal equinox, i.e., the place where the ecliptic intersects the celestial equator, and where the Sun in its apparent motion around the ecliptic crosses over into the northern side of the equator; it is also called the First Point in Aries (the constellation of the Ram). This point has a slow, fairly involved, retrograde motion with respect to the fixed stars, i.e., it moves in the direction opposite to the motion of virtually all bodies in the solar system. Its longitude with respect to a given fixed star decreases by some 50″ per year.

The transformation from the equatorial to the ecliptic system is made with the common x axis pointing toward Q,

$$\begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_0 & \sin\varepsilon_0 \\ 0 & -\sin\varepsilon_0 & \cos\varepsilon_0 \end{pmatrix} \times \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} \quad (4)$$

where ε_0 is the angle between the equator and the ecliptic.

Although many instruments have been constructed since antiquity to measure directly the ecliptic coordinates (see the detailed description of Brahe's instruments by Raeder *et al.*), they are not very exact. Most of the precision measurements of the lunar position are made by timing the Moon's transit day after day by mural quadrants, meridian circles, and other transit instruments, and then transforming to the ecliptic coordinates with the help of the above formulas.

D. The vernal equinox

During antiquity the vernal equinox was in the constellation of Aries, and spring was associated with the Sun's being located at the beginning of the Ram. But since then, the vernal equinox has moved into the "preceding" constellation of Pisces, the Fishes. The astrological literature, however, still associates the constellations in the Zodiac with the twelve periods of the solar cycle as they were in antiquity. Therefore, gentle reader, beware, although you might think of yourself as a Lion, the Sun was in the sign of the Crab when you were born!

The ecliptic is inclined toward the celestial equator by the *obliquity of the ecliptic* ε_0 (really the inclination of the Earth's axis with respect to the ecliptic), about 23°30′. The exact motion of the Earth's axis has a number of periodic terms, collectively called *nutation*, that are of the order of 9″ (see Fedorov, Smith, and Bender, 1980). The ephemerides correct all the geocentric data such as the right ascensions and declinations for the "mean equator" and the "mean equinox of date." The adjective "mean" always applies to changes in some parameter after its periodic variations have been eliminated.

E. All kinds of corrections

For objects in the solar system, a correction is necessary to refer any observations to the center of the Earth, rather than to some location on its surface; the data have to be reduced from the *topocentric* to the *geocentric*. This correction for the parallax is very large for the Moon, almost as much as 1°, whereas for all other objects in the solar system it is of the order of 10″ at most. It changes the apparent position with respect to the fixed stars depending on the distance from the zenith. To make things worse, Zach (1814) showed that the local geology causes the local plumbline to differ significantly from the averaged normal. The shift from topocentric to geocentric coordinates is made by formulas such as Eq. (1) that include the radial distances.

The objects in the solar system have generally a finite size on the order of seconds of arc (and therefore do not twinkle!), except for the Moon and the Sun, whose apparent sizes are very nearly equal and close to 30′ on the average. The Moon's boundary is clearly jagged on the order of seconds of arc because of its mountains; this outline of the lunar shape changes because of the Moon's residual rotation with respect to the Earth.

The bare eye is able to work down to minutes of arc, and requires correction only for refraction and the parallax of the Moon. Isolated observations with a moderate telescope can distinguish features down to seconds of arc that can be seen in good photography. Positions on the celestial sphere to a second of arc, however, require not necessarily large telescopes but very stable and precise mountings so that many data can be taken over extended intervals of time.

F. The measurement of time

The time interval between the transits of the Sun varies considerably throughout the year. The *mean solar day* is defined with the help of a fictitious body called the *mean sun* (MS), which moves on the equator with uniform speed. The difference of the right ascension for the mean sun minus the right ascension for the real Sun is called the *equation of time*, $\alpha_{\mathrm{MS}} - \alpha_S$. This has to be added to the Sun's right ascension α_S to make it increase uniformly.

The hour angle for the mean sun at the Greenwich Observatory is called the *Greenwich mean astronomical time*. *Universal Time* UT is Greenwich mean astronomical time plus 12 hours, so that the transit of the mean sun occurs at 12 hours, and mean midnight at 0 hours. UT is the basis for all the local standard times; they differ from UT simply by adding or subtracting a fixed (not necessarily integer, as in the case of India) number of hours (Howse, 1980).

The difference between the apparent motions of the real and the mean sun is shown on any good sundial. The pointer's shadow at the time of the mean local noon is the *analemma*; it has the shape of an unsymmetric and slanted figure 8 along the north-south direction.

G. The Earth's rotation

The rate of the Earth's rotation decreases because the tides in the oceans act like a brake; the mean solar day increases by roughly 1 millisecond each century. From the vast literature on this subject let me mention the classic study by Newcomb (1878 and 1912) and its modern versions by Marsden and Cameron (1966), Martin (1969), McCarthy and Pilkington (1979), and Babcock and Wilkins (1988). The accumulated lengthening of the day adds up to more than two hours over 20 centuries, or equivalently, the Earth has fallen behind by about 30° in the rotation around its own axis. The reports of eclipses can be used to determine the history of the Earth's rotation. A very detailed account of this argument is found in the books of Robert R. Newton (1976, 1979).

The real trouble with the Earth's rotation is that it decreases sometimes in fits, and occasionally even speeds up again. It was decided, therefore, in the 1950s to use the Earth's orbit around the Sun as the base for the reckoning of time, and to speak about *Ephemeris Time* (ET). The difference between the two times, Universal and Ephemeris, is minute. Ephemeris Time has already been superseded by *Atomic Time* (AT), which is based on atomic clocks, i.e., it is completely independent of the motions in the solar system.

H. The measurement of the solar parallax

For a long time, all measurements of the linear distance between any two objects could be made in only one of two ways: either by directly applying a measuring rod or by triangulation from a base that was short enough to be measured directly with a measuring rod. In this manner Eratosthenes of Cyrene in the early third century B.C. obtained a fair value for the circumference of the Earth. Since the Moon has a large parallax, i.e., the apparent radius of the Earth as seen from the Moon is about 1°, the Greeks concluded correctly that the Moon's distance from the Earth is about 60 Earth radii.

A short treatise by Aristarchus of Samos, also from the early third century B.C., has survived, "On the Sizes and the Distances of the Sun and the Moon," and a translation can be found in Heath (1913). It proposes to determine the exact moment of half moon simply by looking for the time when the lighted portion of the Moon is exactly a half circle. Aristarchus concludes that the ratio of the Sun's distance to the Moon's distance is about 20 (the correct figure is close to 400); in spite of many other efforts to find a better figure, this value was essentially accepted even by Kepler and his contemporaries.

The relative distances for all the planets from the Sun were quite correctly known ever since antiquity. The absolute distances, however, were only obtained at the end of the 17th century by a concerted effort to measure various parallaxes in the solar system, mostly by observing the transits of Mercury and Venus across the face of the Sun from different locations on the Earth. Nevertheless, even Newton and his immediate successors did not have good figures for the solar parallax, and they all used widely different values at different times. That did not impair Newton's theory of universal gravitation, and its application to the solar system, all because of scaling invariance!

I. Scaling in the solar system

Newton derived the complete form of Kepler's third law for the Sun and an isolated planet which attract each other with a force that varies as the inverse square of their distance. In modern notation, let n be the angular frequency ($=2\pi$ divided by the period of the planet's orbit), a the semi-major axis of the planet's orbit, G_0 the gravitational constant (in the appropriate units), and M and m the masses of the Sun and of the planet; then

$$n^3 a^3 = G_0(M+m). \tag{5}$$

Let us now write the equations of motion for an arbitrarily complicated system of mass points that interact only through the gravitational accelerations between them; each mass μ occurs only in the product $G_0\ \mu$. If any solution of these equations has been found, i.e., an explicit expression for each coordinate as a function of time that satisfies the equations of motion, then another solution is obtained by multiplying all the distances with an arbitrary factor κ, provided all the products $G_0\ \mu$ are changed into $\kappa^3\ G_0\ \mu$. If we stick with the same units of time, length, and mass, so that the gravitational constant G_0 keeps the same numerical value, then we effectively assume that each mass has been increased by the factor κ^3.

This scaling preserves the average density of each planet and the Sun; the actual size of a star depends on the interaction of the gravitational forces with forces of a completely different nature. No other forces were known in Newton's time; only rough assumptions could be made, such as the incompressibility of a fluid or the equation of state of an ideal gas. A good value for the solar parallax was secured after the transit of Venus in 1769, and thus the value of the product $G_0\ M$ for the Sun was finally known, but not the value of the mass itself for the Sun, nor for any body in the solar system.

The gravitational constant G_0 was finally measured by Henry Cavendish (1798) with a torsion balance in a famous experiment that measured the gravitational force between two balls made of lead. With G_0 known, the average density of the Earth can be obtained; that was the intention of Cavendish, whose paper is entitled "Experiments to determine the density of the Earth.'' Remarkably, the correct result had been guessed by Newton as about five times the density of water.

Because of scaling invariance, inaccurate values of G_0 affect neither the astronomical observations nor the theory of the motions in the solar system. The observations yield precise values for the mass ratios, and that is all the theory requires. Many of the problems with distances in the solar system were finally solved in the most recent times, particularly in the case of the Moon. On the one hand, space probes circling the Moon in a very low orbit produce a good value for its relative mass directly from Kepler's third law. On the other hand, the reflectors left on the surface of the Moon by the astronauts provide the opportunity to measure directly the distance of the Moon from the Earth through the reflection of laser pulses, giving rise to contemporary ephemerides of unprecedented accuracy. Thus distances have finally replaced angles as the fundamental variables in the description of the Moon's motion.

III. SCIENCE WITHOUT INSTRUMENTS

A. The lunar cycle and prescientific observations

Humanity must have noticed for a long time the Moon's fundamental cycle of about 30 days. Its risings and settings follow a similar pattern in the course of one month to that of the sunrises and sunsets in the course of one year, but their spread on the horizon varies greatly. It reached a minimum in 1997 of only $\pm 24°$ at New York's latitude, much smaller than the Sun's $\pm 31°$; it will increase over the coming nine years and then reach a maximum of $\pm 39°$ in 2006.

The Moon moves about 13° in one day with respect to the fixed stars. When the dark edge of the Moon moves over a star, the apparent width of the light source can be measured very simply by the time interval for its gradual disappearance. (In this manner the apparent width of certain light sources with large redshifts was first established; it was found to be ''quasistellar,'' thereby justifying their name as quasars.)

Other cycles were known in many early cultures. When Sirius first shows up in the dawn's early light at the beginning of August, the Nile is likely to start flooding two weeks later. Venus changes back and forth between evening star and morning star five times in eight years. Jupiter makes a full swing around the Zodiac in about 12 years, more exactly 7 times in 83 years. The cyclic nature of these phenomena probably had a profound effect on religion and philosophy; they were recorded in the early writings of many civilizations. But they cannot be called science as yet because no effort was made to pin down the exact timings and, in particular, the fluctuations around the basic repetitions.

B. Babylonian astronomy

The people living in Mesopotamia, the country in the Middle East between the rivers Tigris and Euphrates, our present-day Iraq, were the first humans to start truly scientific investigations, some time after 1000 B.C. The remaining record is substantial but fragmentary: broken pieces of clay tablets that were found among the ruins of the ancient cities. They were brought to Europe and America by the thousands during the 19th century, not necessarily by archaeologists, and are now kept in places like the British Museum, Columbia, and Yale University.

These clay tablets are densely covered on both sides with letters and numbers that were inscribed with a chisel before the firing of the clay. The interpretation of this ''cuneiform'' writing took many decades of patient effort and is one of the miracles of archaeology. Several hundred tablets are devoted to astronomical pursuits; the oldest of them are from as far back as the 7th century B.C., the time when the city of Babylon rose to the leadership of the Middle East; the most recent are from the 1st century B.C., three centuries after the Greeks under Alexander the Great had conquered that whole part of the world and Babylon had fallen into ruins.

There are extensive diaries covering six to seven months on one tablet reporting the state of the sky from day to day. The data on the Moon include first and last visibility, the stars it passed close by, and the time differences between rising and setting for both the Sun and the Moon around the time of the full moon. These last data are directly related to the apparent speed of the Moon, and bring out one of the basic periods in the lunar motion. Since this kind of analysis is generally recognized as the very first scientific activity of humanity, it is worth describing in more detail.

The sources from this era were finally deciphered at the end of the 19th century by a small group of German Jesuit priests. But the texts are difficult to understand, and the number of experts in this area since then is very small. Neugebauer (1975, 1983) and van der Waerden (1974, 1978) will provide the reader with an introduction to the available literature. As examples of the patient work required in this field and related to the Moon, let me mention Aaboe (1968, 1969, 1971, 1980), Sachs (1974), and Brack-Bernsen (1990, 1993).

C. The precise timing of the full moons

Let us call r, t, and s the times when the Moon rises, transits, and sets; and let us also call r', t', and s' the corresponding times for the Sun. According to a time-honored custom in lunar theory: if z designates any quantity for the Moon, then z' designates the similar quantity for the Sun.

The differences $s'-r$ and $r'-s$ are positive before the full moon, whereas the differences $r-s'$ and $s-r'$ are positive after the full Moon. They are less than one hour around the time of the full moon and can easily be measured with rather primitive water clocks. They were consistently recorded in the diaries, stated in the Babylonian units of time-degree=1/360 of a day =about 4 minutes, and sometimes even further divided into six parts. It is not known whether the recorded times were for first visibility and last appearance on the horizon, or last and first contact with the horizon. The positions of the Moon and the Sun on the horizon are hardly ever directly opposite each other.

Since the differences $s' - r$ and $r' - s$ change sign in the interval of 24 hours, the time of their vanishing can be calculated by assuming their linear decrease. Moreover, the total change of each time difference measures the speed of the Moon relative to the Sun at the time of the opposition.

There is no evidence that the Babylonians had any geometric picture of the events in the sky. But many clay tablets with long and elaborate tables of numbers were found; they are undoubtedly the results of sophisticated numerical models for the lunar and solar cycles, especially of the events near a full moon.

D. The Metonic cycle

Most experts believe that the Babylonians found their procedures for reducing the data simply by staring long enough at long rows of numbers. These data can be generated artificially on the computer nowadays, and one can try to imitate research as it was done by the Babylonians. As a simple example, the reader could look at Goldstine's "New and Full Moons from 1001 B.C. to A.D. 1651," published in 1973.

A 19-year period is found immediately that does extremely well. It manages to predict the full moons consistently within two days; there are 235 lunar cycles during this time so that the average lunar cycle lasts 19 times 365.25 divided by 235=29.53085 mean solar days compared to the modern value of 29.53059 days.

The 19-year cycle became the base for the prevailing calendars in the Middle East; it is generally called the Metonic cycle because it was instituted in Athens as the basis for the Greek calendar in 432 B.C. by the astronomer Meton. The Jewish calendar has a cycle of 19 years with 12 short years of 12 months and 7 long years of 13 months; holidays can therefore be defined with respect to the nearest new moon.

The Christian calendar pays only lip service to the moon with a highly arbitrary division of the year into 12 months whose lengths vary from 28 to 31 days. Nevertheless the 19-year cycle remained alive in Christianity, as can be seen in one of the famous "Books of Hours" from France around 1400 (Zwadlo, 1994) in which the decorative calendar carries symbols to calculate the lunar cycle for any time in the future. The Islamic calendar is based on the opposite decision, i.e., to define the year as 12 lunar cycles, so that 19 years are short of 7 full moons. Therefore the Islamic year-count, which started in +622, is ticking at a faster rate than the Jewish and Christian counts.

E. The Saros cycle

A period of 223 lunations (new moons) emerges from watching both the spread of the moonsets on the horizon and the speed of the Moon near the full moons. Since 235 lunations make 19 years within two hours, the 223 lunations constitute 18 years and 11 days. There are exactly 242 swings on the horizon, while the lunar speed has completed quite exactly 239 cycles. At the same time the Moon has completed exactly 241 trips around the Zodiac.

This new period of 223 lunations in 18 years and 11 days is called the Saros period; it can be recognized very clearly in the sequence of eclipses. The Babylonians kept a careful record especially of the lunar eclipses since they can be observed from many places and are easier to spot. Although the solar eclipses are as frequent, they are harder to notice if they are partial, and total solar eclipses are very rare in any one location. Ptolemy uses three consecutive lunar eclipses from a Babylonian record of the 8th century B.C. in order to compute the parameters for the lunar motions. The Babylonians were not able to make eclipse predictions, but they issued warnings for the eclipses to occur at certain times of the year.

There are many computed lists of lunar and solar eclipses in historical times; the best known and for many purposes quite sufficient are those by von Oppolzer (1887) and Meeus, Grosjean, and Vanderleen (1966), both of them with maps to show the strip of totality in the solar eclipses. The physical significance of the near-periodicities in the Saros cycle have been discussed only very recently by Perozzi, Roy, Stevens, and Valsecchi (1991, 1993).

As if this coincidence of the four periods in the Moon-Earth-Sun system in such a short time span were not enough, the Earth rotates exactly 6585.321 times around its own axis in 223 synodic months. If we wait for three Saros cycles, the Earth will have rotated almost exactly back to its previous orientation, and the eclipse will be seen again from the same place at the same time of the day.

A pair of solar eclipses three Saros cycles apart are of special interest to the author, since he was able to watch the second, whereas the first took place while he was still waiting to be born. Indeed, the meeting of the Dynamical Astronomy Division of the American Astronomical Society (where the author reported his work comparing the lunar calculations of Eckert and Bellesheim with earlier work) was timed to take place at Stanford University at the end of February 1979 so that the participants could repair to Oregon to watch the total eclipse of the Sun on the morning of February 26. Its earlier version had been visible in northern Manhattan and Westchester County in the morning of January 24, 1925. It was the first time that the most recent lunar theory of G. W. Hill and E. W. Brown was used for the prediction. I shall say more of this event in Sec. XI.A.

The Babylonians came tantalizingly near a good theory for their observations. As an example, they had divided the year into two unequal intervals, each roughly one half-year; the Sun was moving at constant speed in each of them and the total ground covered was equal to 360°; the slow speed was given to the interval covering spring and summer, whereas the high speed was given to the fall-winter interval. A smooth sinusoidal curve was approximated by a step function, but the geometric origins, let alone the physics, were never suspected.

IV. THE GOLDEN AGE OF GREEK ASTRONOMY

A. The historical context

To set the stage for this last (but not least) creation of the Greek miracle, one has to be reminded of the time frame in which astronomy came to full flower. After the defeat of the Persians in 479 B.C. under the leadership of Athens and Sparta, the Athenian democracy experienced an economic and cultural explosion that lasted for about 50 years. In 431 B.C., a motley collection of Greek city-states started to grind down the military power of Athens in the Peloponesian war. It ended in 404 B.C. with Athens accepting the terms dictated by Sparta.

The main achievements of this delta function in space and time can still be admired. Scientific astronomy got its start in 432 B.C. when Meton introduced the 19-year cycle into the Athenian calendar. He presumably measured the exact date of this solstice; historians debate whether he knew about the 19-year cycle from the Babylonians or had found it independently.

Greek science in general got its momentum from philosophy in the fourth century B.C. Some of the basic ideas in astronomy seem to have come from Plato and Aristotle, as well as from their students and followers. Meanwhile, Greece had been forcibly unified under King Philip of Macedonia, and his son, Alexander the Great, conquered all of the present-day Middle East including Central Asia and Pakistan. Only then do we encounter the great mathematicians, Euclid and Apollonius, master of the conic sections, as well as Archimedes, in the third century B.C.

Two elementary treatises, ‘‘On the moving sphere’’ and ‘‘On risings and settings,’’ by Autolycus survive from this time. They show that the spherical image of the sky, with its great circles, its daily rotation, and the path of the Sun along the ecliptic was generally accepted; the Greeks had gone way past the Babylonians in finding the geometric underpinnings of astronomy.

Shortly thereafter appeared the paper of Aristarchus of Samos, ‘‘the ancient Copernicus’’ in the words of his translator, Sir Thomas Heath (1913), ‘‘On the Sizes and Distances of the Sun and the Moon’’ (see Sec. II.H). The heliocentric cosmology of Aristarchus was first mentioned by Archimedes in his ‘‘Sand Reckoner,’’ where he tried to estimate the volume and content of the universe (Heath, 1897; see also Dijksterhuis, 1987). Astronomy was bound to take off at this juncture!

And yet, we have to wait one more century for Hipparchus in the middle of the second century B.C., before we get a complete view of astronomy in the modern sense. Unfortunately, only two of his minor works have survived, so that our knowledge of his great accomplishments comes from Ptolemy’s account, another three centuries later, i.e., from the second century A.D.

Hipparchus made the first catalog of about one thousand stars, giving their longitude and latitude and estimating their brightness on a scale from one to six which is still in use, although greatly refined. He described the orbits of the two ‘‘luminaries,’’ Sun and Moon, and the five classical planets, in terms of epicycles, and determined the relevant parameters. He devised a method for predicting the occurrence of solar and lunar eclipses, and he discovered the precession of the equinoxes.

B. The impact on modern science

Greek astronomy managed somehow to grow from the prescientific stage and, with the help of some input from the Babylonian accomplishments, ended up proposing the first complete and scientifically viable picture of the solar system. By the time the Romans took control of the Eastern Mediterranean because the Greeks had been fighting one another for three centuries, astronomy had solid foundations as a science in the modern sense. The idea of first observing and measuring the phenomena in the sky and then predicting future events on the basis of a mathematical model became the goal of all the other sciences that are concerned with the outside world. The description of the solar system has become the basic example of a valid scientific picture. Our intuition and our understanding are still solidly based on the ideas from antiquity.

Astronomy in antiquity had nothing to do with complicated structures of crystalline spheres that moved the solar system around the Earth. The only extant, complete account of ancient Greek astronomy, the *Almagest* of Ptolemy, is a monograph in the modern sense: the various coordinate systems in the sky are first described exactly as we did in the second chapter, along with the relevant mathematics; the theory of the solar motion and of the much more involved lunar motion is explained on purely geometric grounds, and the prediction of the lunar phases and of the two kinds of eclipses is discussed; then follows the study of the fixed stars and the precession of the equinoxes; finally, the apparent motions of the two inferior planets (Mercury, Venus) and of the three superior planets (Mars, Jupiter, Saturn) are treated in the second half of the monograph on the basis of the geocentric universe. An English translation by Taliaferro was published in 1938 as part of Volume 16 of Great Books of the Western World; book-length analyses were undertaken by Pedersen (1974) and Neugebauer (1975).

The history of ancient Greek astronomy has been studied in great detail, and there are many comprehensive accounts for the interested and educated layperson in the principal European languages. The astronomers in the 16th and 17th centuries had studied their Greek, as well as their medieval Islamic and Jewish, ancestors with great care, and some of these works were available in their original as well as in Latin translation. Whereas there is at present a whole industry concerned with the advances during the 16th and 17th centuries, from which we have an abundance of sources quite easily accessible, very few general titles covering Greek and medieval astronomy have been added in this century. Here is an incomplete, but representative, list: Berry (1898), Dreyer (1906), Heath (1913), Dicks (1970), Pedersen and Pihl (1974), Neugebauer (1975).

C. The eccentric motion of the Sun

The Greek astronomers invented different geometric constructions to represent the main results of their observations. We shall look at four examples of the most important models: (i) the eccentric circle for the Sun around the Earth in this section, (ii) the epicycle for the Moon around the Earth in the next section, (iii) the equant for the outer planets in Sec. IV.E, and (iv) the evection for the three bodies, Moon, Earth, and Sun in Sec. V.D. The motion always takes place in the ecliptic, i.e., in a fixed plane, and one special feature in the observed motion is accounted for. The purpose of this exercise is to demonstrate the increasing complexity of these motions and, with the benefit of hindsight, to watch the struggle with a principle of modern mechanics, the conservation of angular momentum.

First, we deal with the slow apparent motion of the Sun in spring and summer, in contrast to its fast motion in fall and winter. The Sun moves with uniform speed on a circle of radius a' whose center is at some distance $\varepsilon' a'$ from the Earth. The two parameters in this model are ε' and the direction of the aphelion (largest distance from Earth), which lies in the direction of the center. They were determined by Hipparchus on the basis of two time intervals, from the vernal equinox to the summer solstice and from the summer solstice to the autumnal equinox. Hipparchus found 1/24 and $65°14'$, in fair agreement with modern values.

If the eccenter model for the Sun's motion is taken at face value, the eccentricity $\varepsilon'=1/24$ would indicate that the Sun's distance from the Earth varies by this amount. The total intensity of the Sun's light would vary by $\pm 2\varepsilon' = \pm 1/12$ These conclusions from the eccenter model for the Sun were not appreciated until Kepler examined the physical consequences of the antique models.

D. The epicycle model of the Moon

This construction is best described if Cartesian coordinates (x,y) are used in the ecliptic, and they are combined into one complex number u,

$$u = x + iy = ae^{i\bar{\lambda}}(1+\varepsilon e^{-i\ell}) = ae^{i\bar{\lambda}} + \varepsilon a e^{i(\bar{\lambda}-\ell)}, \qquad (6)$$

where we have now two angles, the *mean longitude* $\bar{\lambda}$ and the *mean anomaly* ℓ. Here the word anomaly refers to the angle from the apogee to the Moon as seen from the Earth, whereas the longitude is reckoned from the reference point Q; again the adjective mean designates these angles after the elimination of the periodic terms (see Sec. II.D). Each "mean" angle increases linearly with time at its own rate, the mean longitude covering 2π in one *sidereal month* and the mean anomaly covering 2π in one *anomalistic month*, from one apogee to the next. The difference $\bar{\lambda}-\ell$ covers 2π in about nine years.

The first term describes a uniform motion on a large circle around the origin, the *deferent*, while the second term describes the motion around a small circle that rides on the first one, the *epicycle*. Notice that this motion reduces to the eccentric motion of the preceding section when the two angles increase with time at the same rate. The second term on the right then reduces to a translation by the constant εa.

The epicycle construction can also be applied to the motion of the Sun around the Earth. The two angles may not move at the same speed for two reasons: (i) if the solar orbit remains stable with respect to the fixed stars, but the coordinates are fixed to the vernal equinox Q, the Sun then seems to move an additional 50″ each year before it gets to the aphelion; (ii) the aphelion actually moves forward very slowly with respect to the fixed stars; this motion of 12″ per year was discovered by the Islamic astronomers in the early Middle Ages. The aphelion then moves away from the equinox at 62″ per year, and it passed the summer solstice already in A.D. 1250 (see Sec. II.D).

E. The equant model for the outer planets

Both the eccenter and the epicycle motions explain the motion in longitude, but they suffer from a basic flaw that was already noticed at the end of Sec. IV.C: the angular speed and the distance from the Earth vary by the same amount. Ptolemy became aware of this difficulty when he tried the eccenter model for the outer planets, and his remedy turns out to be one of the most ingenious contributions to astronomy.

As the outer planets are watched from the Earth, they move in the forward (Eastern) direction along the ecliptic most of the time, but at regular intervals they reverse their course and go in the Western direction for awhile. The midpoint of this reversal is called the *opposition*, when the outer planet culminates exactly at midnight, i.e., the planet, the Earth, and the Sun lie on one straight line, with the Earth in the middle; the planet is at its brightest because nearest to the Earth. The motion of the outer planet around the Sun can be inferred from the consecutive oppositions, while the varying amplitude of the reversal motion measures directly the relative size of the Earth's orbit compared with the distance of the outer planet.

Ptolemy noticed that the eccenter model for the outer planet, after its eccentricity ε_p had been adjusted to give the correct time intervals between the oppositions, yields twice the observed variation of the reversal motion. Without being aware of the physics involved, he was effectively trying to conserve the angular momentum of the outer planet around the Sun. Here is his ingenious solution of this puzzle.

In the heliocentric picture the outer planet moves on an eccentric circle, whose center O is shifted away from the Sun in S by $\varepsilon_p a'$, but the planet does not run around

O at uniform speed. Rather, an *equant* point S′ is constructed at the distance $2\varepsilon_p a'$ from S, in the same direction as the center O. The planet is made to move at constant angular speed around this equant point S′, rather than around the center O. The outer planet varies its speed on the eccentric circle by twice as much as the variation of its distance from the Sun.

F. The Earth's orbit and Kepler's second law

More than a thousand years were to pass before Ptolemy's clever constructions were seriously questioned. The reader is encouraged to consult the collection of articles by Owen Gingerich (1993) for a more detailed discussion of the development that led from Ptolemy to Copernicus and finally to Kepler. Here is a very brief account.

When Tycho Brahe died in 1600 he left his successor a treasure trove of the most meticulous observations with the best instruments that he was able to build. Kepler studied the orbit of Mars because it has the largest eccentricity among the classical planets (excepting Mercury, which is difficult to observe) and could be expected to yield the most telling clues for the renewal of astronomy. But before starting this challenging task, he had to know exactly the orbit of the Earth around the Sun because, after all, that was the base for Brahe's data. Almost one-fourth of Kepler's *New Astronomy*, published in 1609, is devoted to this preliminary project. Max Caspar (1929), the editor of Kepler's Collected Works, has published a beautiful translation of the *New Astronomy* into German. Aside from a detailed paraphrase by Small (1963), English speakers had to wait until 1992 for a translation by Donahue (1992).

Kepler triangulates the Earth's orbit with the help of a fixed base, for which he uses the position of Mars at regular intervals of 687 days, the period of Mars in its orbit around the Sun S. He is tremendously pleased with his discovery that the equant model is also applicable to the Earth E. First he argues that the Earth's speed near the perihelion P and the aphelion A is inversely proportional to the distance r' from the Sun. Since he is convinced that the Earth's motion is determined by the Sun, he then generalizes this idea: if the Earth's orbit, say the eccentric circle, is broken up into short intervals, the total time to get from A to E would be proportional to the sum over all these short intervals where each is multiplied with its distance from S.

Although Kepler takes the trouble to show that this way of calculating the earth's motion differs only insignificantly from the equant construction, he finds this sum very cumbersome to compute. He now searches for an easier way of relating his construction by small intervals to the real Sun in S. He hits upon his second law; the time for the Earth to get from A to E is proportional to the area that is swept out by the vector from S to E. He shows that this third way of computing is consistent with the two other models for the Earth.

G. The elliptic orbit of Mars

Kepler now faces the main part of his epic struggle, namely, to find the exact orbit of Mars around the Sun. The accurate knowledge of the Earth's orbit around the Sun and Brahe's observations over 20 years show that the orbit of Mars is not an eccentric circle. The *New Astronomy* gives a detailed report of all the detours and lucky incidents that finally led to the idea of the ellipse with the Sun in one of the foci, and the times proportional to the area.

The role of the mean anomaly is taken over by the area that is swept out by the radius from the Sun to Mars, counted from the perihelion. If this area is normalized to 2π for one complete orbit, it is represented by the angle ℓ', whose value increases linearly with time. The true *anomally* f' and the *eccentric anomaly* v' keep their original meaning as the angles seen from the Sun and from the center of the orbit, all of them measured from the perihelion.

The mathematical relations between these angles and with the distance from the Sun turn out to be elementary, but still quite tricky. Leaving out the primes on the following formulas, one gets for the distance

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos f}, \tag{7}$$

where a is the semi-major axis and ε the eccentricity. In Cartesian coordinates with the x axis in the direction of the perihelion, the ellipse is given by the equations

$$x = r\cos f = a(\cos v - \varepsilon),$$
$$y = r\sin f = a\sqrt{1-\varepsilon^2}\sin v. \tag{8}$$

The connection with the mean anomaly ℓ is given by *Kepler's equation*,

$$v - \varepsilon\sin v = \ell = n(t-t_0), \tag{9}$$

where t_0 is the time of perihelion passage, and the *mean motion* n is the mean angular speed, 2π divided by the period T_0.

H. Expansions in powers of the eccentricity

The eccentric anomaly v is of no interest, but it cannot be avoided when the radius r and the true anomaly f are directly expressed in terms of the mean anomaly ℓ, i.e., the time t and the eccentricity ε. The relevant expressions can only be given as Fourier expansions in ℓ whose coefficients are power expansions in ε. These expansions are not hard to get; but when the astronomer Friedrich Wilhelm Bessel tried to compute them to high order, he found it necessary to invent the functions that now carry his name! Notice that the mean anomaly ℓ is counted from perihelion, a convention that we shall keep from now on.

Only the terms to order ε^2 will be listed:

$$\frac{r}{a} = 1 + \frac{1}{2}\varepsilon^2 - \varepsilon\cos\ell - \frac{1}{2}\varepsilon^2\cos 2\ell + \cdots, \tag{10}$$

$$f = \ell + 2\varepsilon \sin \ell + \frac{5}{4} \varepsilon^2 \sin 2\ell + \cdots . \quad (11)$$

These formulas will eventually reappear when the motion of the Earth around the Sun is taken into account explicitly to get the corrections for the motion of the Moon around the Earth.

Finally, we list two more expressions for the same expansions in order to make the connection with the epicycle theory,

$$\frac{1}{a}(x+iy) = \frac{r}{a} e^{if} = e^{i\ell}\Big[1 - \frac{1}{2}\varepsilon^2 - \frac{3}{2}\varepsilon e^{-i\ell} + \frac{1}{2}\varepsilon e^{i\ell} + \frac{1}{8}\varepsilon^2 e^{-2i\ell} + \frac{3}{8}\varepsilon^2 e^{2i\ell} + \cdots\Big]. \quad (12)$$

The simple epicycle idea has become a Fourier expansion. Equivalently,

$$\frac{1}{a}(x+iy) = \frac{r}{a} e^{if} = e^{i\ell}\Big[1 - \frac{1}{2}\varepsilon^2 - \varepsilon \cos \ell + 2i\varepsilon \sin \ell + \frac{1}{2}\varepsilon^2 \cos 2\ell + \frac{i}{4}\varepsilon^2 \sin 2\ell + \cdots\Big], \quad (13)$$

where the real part indicates the correction in the distance r, while the imaginary part describes the correction in the true anomaly f. In lowest order, the latter is obviously twice as large as the former. The factor 2, between $-\varepsilon \cos \ell$ and $2i\varepsilon \sin \ell$, is designed to preserve the angular momentum and was correctly given in Ptolemy's equant model.

V. THE MANY MOTIONS OF THE MOON

A. The traditional model of the Moon

A plane through the center of the Earth is determined at an inclination γ of about 5 degrees with respect to the ecliptic. The Moon moves around the Earth in that plane on an ellipse with fixed semi-major axis a and eccentricity ε of about 1/18. The Greek model was quite similar, except that the ellipse was replaced by an eccentric circle.

The plane itself rotates once every 18 years in the backward direction, i.e., against the prevailing motion in the solar system, while keeping its inclination constant. The perigee of the Moon, its point of closest approach to the Earth, makes a complete turn in the forward direction in about nine years.

The following picture (see Fig. 1) emerges: first we fix the direction of the spring equinox or some fixed star near it as the universal reference Q in the ecliptic: counting always from west to east, we determine the angle h from Q to the ascending node, i.e., the line of intersection for the Moon's orbit with the ecliptic where the Moon enters the upper side of the ecliptic; from there we move by an angle g in the Moon's orbital plane until we meet the perigee of the Moon; and finally we get to the Moon by moving through the true anomaly f. All these three angles have a double time dependence: linear (increasing for f and g, while decreasing for h) plus various periodic terms that average to 0.

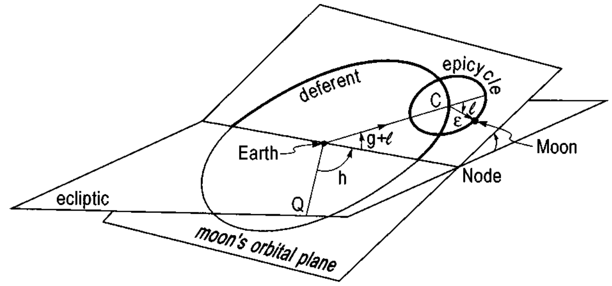

FIG. 1. The basic lunar model from antiquity (adopted ever since) consists of an orbital plane for the Moon containing an epicycle for its orbit; the crucial parameters a,ε,γ and the three angles l,g,h have their modern interpretation.

B. The osculating elements

Assuming that the Moon's position $\vec{x}$ and its momentum $\vec{p}$ with respect to the Earth are known at some time t, its total energy (kinetic plus potential with respect to the Earth) gives the semi-major axis a; its angular momentum $\vec{L}$ yields not only the inclination γ and orientation h of its orbital plane, but also its eccentricity ε; finally, its so-called Runge-Lenz vector,

$$\vec{F} = [\vec{p}, \vec{L}] + G_0 E M^2 \vec{x}/r, \quad (14)$$

gives the location of the perigee, i.e., the angle g with respect to the node, and from there the true anomaly f. The masses of the Earth and of the Moon are called E and M, while G_0 is the gravitational constant.

These elements a, ε, γ, h, g, f for the Moon give the parameters of the Kepler ellipse that fits the lunar trajectory most closely at the time t; they have complicated variations with the time t. The linearly increasing parts in the angles get special names and symbols; they are used as the basis for all the future computations. The mean anomaly ℓ is the linearly varying part of the true anomaly f; the *mean argument of the latitude F* is the linear part of the distance $f+g$ from the node; the mean longitude λ is the linear part of the distance $f+g+h$ from the reference Q.

Whereas a single variable, the mean longitude, is sufficient for describing the complete motion of a planet around the Sun, three angles are required for the Moon: the mean longitude λ for the main motion around the Earth, the mean argument of latitude F for the motion out of the ecliptic, and the mean anomaly ℓ for the radial motion. The osculating elements a, γ, and ε have only periodic variations. But, according to Fig. 2, it is quite misleading to think of the lunar orbit as having a fixed eccentricity like $\varepsilon \cong 1/18$, because the value of the osculating eccentricity varies enormously and quite fast.

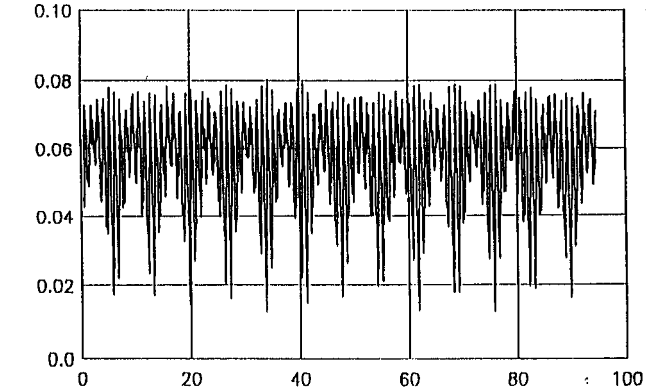

FIG. 2. The effective eccentricity of the lunar trajectory as a function of time; the abscissa gives the time in synodic months, starting with the year 1980; the traditional picture of the Moon's motion is obviously not adequate (see Gutzwiller, 1990, page 60).

C. The lunar periods and Kepler's third law

To each rate of change for an angle with linear time dependence corresponds a period that was well known to the Babylonians and Greeks:

T_1=tropical month (equinox to equinox)=27.32158 days,

T_2=anomalistic month (perigee to perigee)=27.55455 days,

T_3=draconitic month (node to node)=27.21222 days,

measured in mean solar days. The fifth decimal corresponds to 1 second of time and was correctly known to the Greeks. The mean longitude λ increases by 2π in the period T_1, the mean anomaly ℓ increases by 2π in the period T_2, and the argument of latitude F in the period T_3.

By combining the tropical month with the period for the Sun's returning to the spring equinox,

$$T_0=\text{tropical year}=365.2422\ \text{days},$$

we obtain the second most familiar period in this system, the average time between new moons, which turns out to be $T=T_0T_1/(T_0-T_1)=29.53059$ days. The *mean elongation* $D=\lambda-\lambda'$ increases by 2π in one synodic month.

The Greeks related all events in the sky to the spring equinox, but an inertial system of reference is preferable when doing physics. The return of the Moon and the Sun to the same fixed star defines the sidereal month $T_1=27.32166$ days and the sidereal year $T_0=365.257$ days.

Let n be the rate of increase of the Moon's mean longitude λ, and n' the rate of increase of the Sun's mean longitude λ'. Then Kepler's third laws are, in the complete form that was first given by Newton,

$$n^2a^3=G_0(E+M),\quad n'^2a'^3=G_0(S+E+M). \tag{15}$$

Only the products G_0M, G_0E, and G_0S appear in the following discussion. The semi-major axes a for the Moon and a' for the Sun can be regarded as defined in terms of the mean motions n and n' by Eqs. (15), or the other way around.

D. The evection—Greek science versus Babylonian astrology

The Babylonians knew that the full moons could be as much as 10 hours early or 10 hours late; this is due to the eccentricity ε of the Moon's orbit. But the Greeks wanted to know whether the Moon displays the same kind of speedups and delays in the half moons, either waxing or waning. The answer is found with the help of a simple instrument that measures the angle between the Moon and the Sun as seen from the Earth.

The half moons can be as much as 15 hours early or late. With the Moon moving at an average speed of slightly more than 30′ per hour (its own apparent diameter!), it may be as much as 5° ahead or behind in the new/full moons; but in the half moons, it may be as much as 7°30′ ahead or behind its average motion. This new feature is known as *evection.*

Ptolemy found a mechanical analog for this peculiar complication, called the crank model. It describes the angular coupling between Sun and Moon correctly, but it has the absurd consequence of causing the distance of the Moon from the Earth to vary by almost a factor of 2.

In the thirteenth century Hulagu Khan, a grandson of Genghis Khan, asked his vizier, the Persian all-round genius Nasir ed-din al Tusi, to build a magnificent observatory in Meragha, Persia, and write up what was known in astronomy at that time. Ptolemy's explanation of the evection was revised in the process. In the fourteenth century Levi ben Gerson of Avignon in southern France seems to have been the first astronomer to measure the apparent diameter of the Moon (see Goldstein, 1972, 1997). Shortly thereafter Ibn al-Shatir of Damascus in Syria proposed a model for the Moon's motion that coincides with the theory of Copernicus two centuries later. The crank model was replaced by two additional epicycles, yielding a more elaborate Fourier expansion in our modern terminology (see Swerdlow and Neugebauer, 1984).

With the improvements of the Persian, Jewish, and Arab astronomers, as well as Copernicus, the changes in the Moon's apparent diameter are still too large with $\pm 10\%$. As in Kepler's second law, the Fourier expansion (12) has to include epicycles both in the backward and in the forward direction, in the ratio 3:1. Thus one eventually finds for $u=x+iy$ the expansion

$$\begin{aligned} &ae^{iD}\left(1-\frac{3}{2}\,\varepsilon e^{-i\ell}+\frac{1}{2}\,\varepsilon e^{+i\ell}+\frac{1}{2}\,\delta e^{+2iD-i\ell}\right.\\ &\qquad\left.-\frac{3}{2}\,\delta e^{-2iD+i\ell}\right)\\ &=ae^{iD}(1-\varepsilon\cos\ell-\delta\cos(2D-\ell)\\ &\quad+2i\varepsilon\sin\ell+2i\delta\sin(2D-\ell)). \end{aligned}\tag{16}$$

The second line shows the maximum deviation from the uniform angular motion to vary between $\pm 2(\varepsilon-\delta)$ in the full/new moons and $\pm 2(\varepsilon+\delta)$ in the half moons. Therefore we finally get $\varepsilon = .055$ and $\delta = .011$. The distance of the Moon varies at most by $\pm 6.6\%$, and the apparent diameter of the Moon varies between $28'$ and $32'$.

E. The variation

At the end of the 16th century, Tycho Brahe measured continuously for over twenty years everything that happened in the sky. He was lucky because he witnessed in 1572 the last big supernova in our Galaxy (the supernova of 1604 seen by Kepler was much smaller), and we have been waiting for any supernova in our galaxy ever since. He also saw a large comet in 1577 and was able to show that it was outside the sphere of the Moon. He did all that with his bare eyes and those of his assistants, after he had built the largest and best instruments ever, and reached a precision of 1 minute of arc. He published a detailed description of his magnificent observatory on the Danish island Hveen (see the English translation of Raeder, Stroemgren, and Stroemgren, 1946).

Brahe's lunar theory is contained in Part 1 of his *Preparatory School for the New Astronomy*, which was edited by Kepler and published posthumously in 1602 (Volume 2 of the *Complete Works*). Dreyer's biography (1890) contains a short discussion of Brahe's scientific work, whereas Gade's (1947) is concerned with his turbulent life; Thoren provides a special chapter on the theory of the Moon in *The Lord of Uraniborg* (Thoren, 1990).

Four new "inequalities," i.e., periodic deviations from the uniform motion of the Moon around the Earth, were discovered by Brahe. Kepler tried to give all these motions a physical interpretation, on the strength of his boundless imagination and without trying to figure out quantitatively how large they are.

The most interesting of Brahe's lunar discoveries is the *variation*, a not particularly informative name that has caused some confusion. It is the third largest correction to the longitude of the Moon: the anomaly causes deviations from the mean longitude up to $6°15'$, the evection adds another $1°15'$, while the variation accounts for a further $40'$. It depends on twice the mean elongation $D = \lambda - \lambda'$, i.e., twice the mean angular distance of the Moon from the Sun. The variation plays a crucial role in Hill's theory of the Moon.

F. Three more inequalities of Tycho Brahe

Brahe also found the "annual, inequality" with an amplitude of $11'$, which slows down the Moon in its motion around the Earth when the Earth is near perihelion and speeds it up near aphelion.

Finally, Brahe found that the Moon's orbital plane can be best understood by its vertical direction's describing a small cone around the vertical to the ecliptic. The opening angle of this cone is $5°11'$, and the motion of the vertical around the cone takes about 18 years. Brahe recognized that this motion is not quite uniform, but acts like a deferent with a small epicycle of radius $9'$ that turns twice every synodic month. It is as if the Moon's orbital plane straightens up a bit when facing the Sun, which is reasonable on physical grounds.

The net effect on the lunar latitude is similar to the evection: the main term, $\gamma \sin(F)$ (see Sec. V.B), has to be corrected by a term proportional to $\sin(F-2D)$ (see Sec. V.D) with an amplitude of $-9'$. It looks like the axis of a gyroscope oscillating while going around its fixed cone. It is accompanied by a periodic shift of the line of nodes that amounts to $96'$, a phenomenon that is usually called a libration.

By the middle of the 17th century, the astronomy of the solar system had reached a point where any further progress along the same lines could only confuse the new picture of the universe that Copernicus, Brahe, and Kepler had created. With the help of the telescope there was no lack of serious effort; observational methods improved rapidly and led to an accumulation of new data that needed more than just an increasing number of empirical parameters. A first systematic survey of the astronomical observations during the 17th century was made by Pingré, but was published only in 1901. Meanwhile Newcomb (1878) had collected all the relevant lunar data to 1900. Gingerich and Welther (1983) have compared astronomical tables from the 17th century with modern computations. The situation in lunar theory then had some similarities with our present conditions in nuclear and high-energy physics, where we seem to be drowning in a flood of first-class data without a simple and efficient theory that can be generally understood while giving good quantitative results. Of course, the man to change all that for the Moon's motion was Isaac Newton.

VI. NEWTON'S WORK IN LUNAR THEORY

A. Short biography

By the middle of the 17th century the passion for science had grown to such an extent that it became an official function of the state. In 1666, under the leadership of the young Louis XIV and his prime minister Colbert, the French government organized its Royal Academy, where Christiaan Huygens became the best paid member at the age of 37. In his efforts to build more reliable clocks, he discovered and then published in 1673 the law of circular motion: *the centrifugal force is proportional to the mass and the square of the velocity, and inverse to the radius*. He also visited England several times and participated in experiments to establish the laws of motion for bodies impacting on one another (see Bell, 1947).

Meanwhile, an obscure fellow at the University of Cambridge had figured out the same laws of mechanics on the basis of some exceedingly clever arguments such as bouncing a body inside a square box to get the centrifugal force. Isaac Newton, born on Christmas Day of

1642 (the year Galileo died), had done entirely novel work in mathematics, mechanics, and optics while a student and then a fellow at Cambridge, but hardly anybody knew about it. In 1669, he was lucky to become the Lucasian professor of mathematics upon the resignation of the more ambitious Isaac Barrow. Newton acquired fame, and membership in the recently founded Royal Society of London, with his work in optics and the construction of his reflecting telescope in 1671. But the publicity and the scientific arguments were too much for him, and he almost completely retired from contact with any colleagues for over a decade, while he devoted his time mostly to theology and alchemy.

His scientific talents were finally mobilized again when the new secretary of the Royal Society, the astronomer Edmond Halley, went to see Newton in August 1684 to get the answer to an important problem: If Huygens' formula for the centrifugal force is combined with Kepler's third law, the force that attracts the planets to the Sun or the Galilean moons to Jupiter is found to vary inversely with the square of the distance. Halley had come to this conclusion in conversations with the physicist Robert Hooke and the engineer-architect Christopher Wren at a meeting of the Royal Society in January 1684. But none of them had been able to show that Kepler's first and second laws could be derived directly from the assumption of this inverse-square-of-the-distance force. Newton claimed that he had derived Kepler's laws in this way some time ago, but he could not find the relevant papers.

In November 1684 Halley received a manuscript from Newton, "On the Motion of Bodies in an Orbit," that contains all we know now about the motion on conic sections, including even a discussion of the effect of a resisting medium (see Ball, 1983; Hall and Hall, 1962; Herivel, 1965; *Mathematical Papers*, 1967–1981; Cohen, 1978). Two years later, the printing of *The Mathematical Principles of Natural Philosophy* (usually referred to as the *Principia* from the Latin title) got started under Halley's watchful eye, and on July 5, 1687 the task was completed (Newton, 1687). This monumental work has 510 pages of tightly argued mathematical physics, all of them conceived and written in two and a half years. The two-page preface gives Halley some credit for his "encouragement and entreaties" to publish and offers the following remarks on Newton's work concerning the Moon:

"*But after I had begun to consider the inequalities of the lunar motions, . . . , I deferred that publication till I had made a search into those matters, and could put forth the whole together. What relates to the lunar motions (being imperfect), I have put all together in the corollaries of Proposition LXVI, ... Some things, found out after the rest, I chose to insert in places less suitable, rather than change the number of propositions and the citations*." Among the few subjects that Newton mentions in the preface to the second edition in 1713, he says that "*the lunar theory and the precession of the equinoxes were more fully deduced from their principles*." And in the equally short preface to the third and last edition of 1726, "*the argument to prove that the moon is retained in her orbit by the force of gravity is more fully stated*." It sounds almost like "back to the drawing board."

In the years after the completion of the *Principia* in 1687, Newton did some work in mathematics and optics; he also went back to alchemy, but he hardly touched the topics in the *Principia* any more. He tried to get better observations of the Moon from the Astronomer Royal, John Flamsteed, who was more interested in establishing his great star catalog, and the two parted ways after quarreling.

Newton left the pursuit of science altogether after his breakdown in 1694. He became Warden and then Master of the Mint, a job which he pursued with great energy and in which he essentially controlled the circulation of money in the United Kingdom. He became wealthy, but his scientific life was limited to presiding over the Royal Society with an iron hand until his death in 1728. His pivotal role in the development of mathematics, astronomy, and physics is all concentrated in relatively few years of his long life.

Many biographies of Newton have been written, especially in the last decades, for example, those of, Manuel (1968), Westfall (1980), Christianson (1984), Gjertsen (1986), and White (1997). There are always some new documents to be discussed and circumstances to be noted that were not appreciated in earlier reports. One of the first biographies was written by Sir David Brewster of optics fame; it has the great virtue of pursuing Newton's achievements and their further developments all the way to 1855 when it was published. Although many documents concerning Newton's life and work have been discussed at great length ever since, he remains a lonely and mysterious figure with achievements to his credit that have no equal in the history of science.

B. *Philosophiae Naturalis Principia Mathematica*

Newton's monumental work is hard to understand even in a modern translation. His great predecessors, Galileo, Kepler, and Huygens, are generally easier to digest, perhaps because their achievements are simpler, but also because they tried to explain themselves to lesser mortals. Whereas the reader of a modern monograph is overwhelmed with intricate technical details, Newton presents a sequence of statements about the geometric relationships in diagrams of deceptive simplicity.

After two short introductory chapters entitled "Definitions" and "Axioms or Laws of Motion," Newton moves immediately to the core of the matter. Proposition I is the statement of Kepler's second law for the general case of a centripetal force, i.e., an attractive force that depends only on the distance from the center. The sixth corollary of proposition IV gives credit to Halley, Hooke, and Wren, and various problems are solved in the same section. Then comes the important proposition XI, in which the central force in a Kepler ellipse is shown to vary as the inverse square of the distance. The same thing is shown for hyperbolas and parabolas, and proposition XV then proves Kepler's third law, on page 56 of a volume with 510 pages.

The remainder of Book I covers many problems that are connected with orbits whose shape is a conic section, e.g., finding the Kepler ellipse from a certain set of initial conditions. The first statements are made that are equivalent to the conservation of energy. Section 9 will be discussed in detail below because it deals with trajectories that result from rotating a stationary orbit. Section 11 treats two-body problems without external forces and derives Kepler's third law in its complete form [Eq. (15)] which is required for double stars. This section ends with proposition LXVI, which is central for lunar theory. Section 12 covers the potential theory for bodies of spherical shape, and section 13 does it for bodies of arbitrary, in particular ellipsoidal, shapes. The comparison with astronomical observations is left to Book III (the last) entitled "The System of the World," where many of the earlier results are clarified.

There exists a small volume entitled *A New and most Accurate Theory of the Moon's Motion* "whereby all her Irregularities may be solved, and her Place truly calculated to Two Minutes. Written by that Incomparable Mathematician Mr. Isaac Newton, and published in Latin by Mr. David Gregory in his Excellent Astronomy. London, Printed and sold by A. Baldwin in Warwick-Lane. 1702." It is more a description than an explanation such as Newton attempted in the *Principia*. It contains, however, a very eloquent statement of Newton's attitude toward lunar theory. The preface "To the reader" starts out: "*The Irregularity of the Moon's Motion hath been all along the just Complaint of Astronomers; and indeed I have always look'd upon it as a great Misfortune that a Planet so near us as the Moon is, . . . should have her Orbit so unaccountably various, that it is in a manner vain to depend on any calculation . . . , though never so accurately made*." Notice the marvelous definition of what we would call chaos nowadays! See Cohen (1975), as well as Waff (1976) and (1977).

A number of major as well as minor scientists have taken the trouble to present Newton's arguments for students in the sciences and for the educated layperson. Outstanding among these authors are the astronomers Sir George Airy in 1834 and Sir John Herschel (son of William, the discoverer of Uranus) in 1849, as well as Henry Lord Brougham (formerly Lord Chancellor) and E. J. Routh, who wrote a very useful *Analytical View of Sir Isaac Newton's Principia* in 1855. Two recent books make a valiant effort to introduce the reader to the unfamiliar methods of the *Principia:* Brackenridge (1995) and Densmore (1995) cover only the proofs of Kepler's laws whereas Chandrasekhar (1995) discusses both Books I and III; see the essay review by Westfall in *Isis* (1996).

Lunar theory was the first instance in all the sciences where sheer intuition was no longer sufficient to keep up with the rapidly increasing accuracy of the observations. Nevertheless, we shall track down a few of Newton's ideas on this subject to see how far he was able to get, although he abandoned his search while still far from his own goals.

C. The rotating Kepler ellipse

Lunar theory makes its first surreptitious appearance in propositions 43 to 45, where Newton discusses the possibility of a Kepler ellipse rotating at some uniform rate as it occurs for the Moon. He assumes a purely centripetal force, so that Kepler's second law is still valid. The apsides (places of largest and smallest separation) rotate because the force is no longer assumed to be inversely proportional to the square of the distance r. The resulting trajectory looks like the petals of a flower.

Let us set up the problem in the way in which it would appear in a modern textbook. A body of known mass m_0 is moving around a fixed center, which is located at the origin of a polar coordinate system (r,ϕ). The force is a known function $F(r)=-dV/dr$ in terms of the potential $V(r)$. Newton introduces the idea of a conserved angular momentum L without defining any such quantity. The equation for the radial motion becomes

$$m_0 \frac{d^2r}{dt^2}=F(r)+\frac{L^2}{m_0 r^3}. \tag{17}$$

Newton's proposition 44 says essentially the same in words.

The eccentricity ε is assumed to be small, so that the orbit is very close to a circle, and the distance between the two bodies varies only over a narrow range (r_1,r_2). Newton now comes up with an ingenious trick that reveals his deep understanding of physics, although the reader has to figure out in her own terms what exactly goes on in Newton's mind: Since the $1/r^2$ force works out so well, whatever the centrifugal force that goes as $1/r^3$, why not make an expansion for the arbitrary force $F(r)$ where only these two terms occur?

Newton treats as his second example the case in which $F(r)=-\text{const}\, r^\nu/r^3$, where the exponent $\nu>0$, and $\nu=1$ for the gravitational force. The change Ψ in the angle ϕ from one closest approach to the next becomes $2\pi/\sqrt{\nu}$, which is the main result of Newton's section 9 after an 11-page argument.

D. The advance of the lunar apsides

Without explaining what he is doing, Newton proposes as the third example of the advancing apsides the case of a small perturbative force that is repulsive and varies linearly with the distance, so that

$$F(r)=-\frac{G_0M_0m_0}{r^2}+m_0\omega^2 r, \tag{18}$$

where the circular frequency ω is our parameter to keep the correct dimensions of a force. Its value remains open at this point.

The calculation follows the same pattern as in the preceding section and yields

PROPOSITIO LXVI. THEOREMA XXVI.

Si corpora tria, quorum vires decreſcunt in duplicata ratione diſtantiarum, ſe mutuo trahant; & attractiones acceleratrices binorum quorumcunque in tertium ſint inter ſe reciproce ut quadrata diſtantiarum; minora autem circa maximum revolvantur: dico quod interius circa intimum & maximum, radiis ad ipſum ductis, deſcribet areas temporibus magis proportionales, & figuram ad formam ellipſeos umbilicum in concurſu radiorum habentis magis accedentem; ſi corpus maximum his attractionibus agitetur; quam ſi maximum illud vel a minoribus non attractum quieſcat, vel multo minus vel multo magis attractum, aut multo minus aut multo magis agitetur.

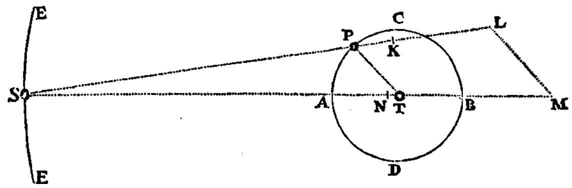

FIG. 3. Newton's Proposition LXVI from the first book of the *Principia* explains the essence of his lunar theory (see the quotation in Sec. VI.A); its 22 corollaries cover 15 pages, and contain a single figure, which is repeated on every second page in the third edition. The Sun's attraction to the Moon is represented by the line SL and is decomposed into the sum $SM + ML$ for the detailed discussion.

$$\Psi \cong 2\pi\left(1+\frac{3\omega^2}{2n^2}\right), \tag{19}$$

provided the second term is small. In the case of the Moon, the angular speed is given by $n=2\pi/T_1$, where T_1 is the sidereal month.

Without any explanation, Newton now proposes for the ratio ω^2/n^2 the value 100/35745 and concludes that the perigee advances by $1°31'28''$. Only the third edition of the Principia, almost 40 years later, ends this whole section with the cryptic remark: "*The apse of the Moon is about twice as swift.*" Nevertheless, after the data had been known with high accuracy for over 2000 years, Newton was able to reduce the difference between the anomalistic and the sidereal month to the ratio $(T_1/T_0)^2$. The missing factor 2 in the motion of the Moon's perigee was finally explained 20 years after his death.

E. Proposition LXVI and its 22 corollaries

Modern language and contemporary symbols like vectors will be used in this section to explain the basic ideas in proposition LXVI. Nevertheless, it is a sacred duty to emphasize what we owe to Newton's crucial insights, without which there would be no progress in understanding the motion of three masses that interact through gravitational forces. All his reasoning over 15 pages is based on a single diagram (Fig. 3), which is repeated seven times, on every second page, to spare the reader the tedium of turning pages while thinking. The corresponding Fig. 4 will be used as the modern equivalent, where the nomenclature agrees with the later chapters.

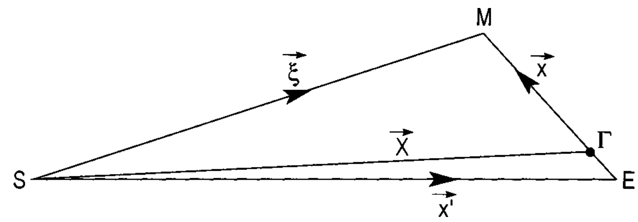

FIG. 4. Modern version of Fig. 3 containing the center of mass Γ of the Earth-Moon double planet that is shown to circle the Sun almost exactly on a Kepler ellipse. The vectors are $\vec{x}$ from the Earth in E to the Moon in M, $\vec{X}$ from the Sun in S to Γ, $\vec{x}'$ from S to E, and $\vec{\xi}$ from S to M.

The main proposition as applied to the lunar problem claims the following: *The motion of the Moon around the Earth under the gravitational attraction of both the Earth and the Sun comes closer to the ideal Kepler motion, if the Earth in turn is allowed to move under the gravitational attraction of both the Sun and the Moon rather than being forced to remain at a fixed position.*

In Fig. 3 we have the Earth with mass E in T (tellus), the Sun with mass S in S (sol), and the Moon with mass M in P (planeta). In Fig. 4 the Moon's and Earth's locations are designated by M and E, while Γ is the center of mass for the double planet Earth-Moon. The vectors connecting various points in Fig. 4 are $\vec{x}$ of length r pointing from E to M,$\vec{x}'$ of length r' pointing from S to E, ξ of length ρ pointing from S to M, and X of length R pointing from S to Γ.

The crux of Newton's argument lies in representing the "accelerative force" of the Sun on the Moon by the segment SL in Fig. 3. This segment is then decomposed into the sum of the two segments SM and ML, where ML is parallel to the line TP which connects the Earth with the Moon. Referring to Fig. 4, we get the first component of the Moon's perturbation by the Sun,

$$G_0 S\vec{x}'\left(\frac{1}{\rho^3}-\frac{1}{r'^3}\right)\cong 3G_0 S\,\frac{\vec{x}'}{r'}\,\frac{(\vec{x}',\vec{x})}{r'^4}, \tag{20}$$

which is clearly of the same order as the second component,

$$-G_0 S\,\frac{\vec{x}}{\rho^3}\cong -G_0 S\,\frac{\vec{x}}{r'^3}. \tag{21}$$

Newton now studies the effect of these perturbative accelerations on the orbit of the Moon around the Earth. According to Corollary 2, the Moon's angular momentum with respect to the Earth is larger in the syzygies (new and full moon) than in the quadratures (half moons). Moreover, according to Corollary 3, the Moon's speed is larger in the syzygies than in the quadratures. Corollary 4 points out, however, that the lunar orbit is more highly curved in the quadratures than in the syzygies because the solar perturbation decreases the Earth's attraction in the syzygies, while its radial component helps in the quadratures. Corollary 5 explains that the Moon is further from the Earth in the quadratures than in the syzygies. But we still have to account for the motion of the apsides.

F. The motion of the perigee and the node

From Corollary 7 onward, Newton gets to the main job of estimating the strength of the solar perturbation. Since $r' \cong a'$, i.e., the semi-major axis of the Earth's orbit around the Sun, one can simplify Eqs. (20) and (21) by approximating,

$$\frac{G_0 S}{r'^3} \cong \frac{G_0 S}{a'^3} = n'^2, \tag{22}$$

with the help of Kepler's third law [Eq. (14)]. The first component [Eq. (20)] of the solar perturbation is always repulsive, and its average over all directions reduces to $+(3/2)n'^2 r$, while the second component [Eq. (21)] becomes simply $-n'^2 r$. If these two contributions are added and multiplied with the Moon's mass M, the averaged force of the Sun on the Moon becomes $+(M/2)n'^2 r$.

In order to apply the result from Sec. VI.D, we have to set $m_0 = M$ and $\omega^2 = n'^2/2$ in Eqs. (18) and (19). Since $(n'/n)^2 = 1/178.7$ is exactly twice the ratio 100/35745 which was used by Newton, we find the explanation for his cryptic statement. His result can also be written in the form.

$$n_2 = \frac{2\pi}{T_2} \cong n\left(1 - \frac{3}{4}\frac{n'^2}{n^2}\right), \quad T_2 \cong T_1\left(1 + \frac{3n'^2}{4n^2}\right). \tag{23}$$

Newton expands the scope of his arguments with the help of some striking images in the later corollaries. The mass of the Moon is distributed in a rotating ring around the Earth. This "lunar ring" is inclined by γ with respect to the ecliptic and experiences the pull of the Sun in the syzygies. Nowadays we would conceive of the lunar orbit as a fast gyroscope whose axis of rotation is not quite perpendicular to the ecliptic. The change in angular momentum is due to the solar torque and yields the regression of the lunar node,

$$n_3 = \frac{2\pi}{T_3} = n\left(1 + \frac{3}{4}\frac{n'^2}{n^2}\right), \quad T_3 = T_1\left(1 - \frac{3n'^2}{4n^2}\right). \tag{24}$$

The 22 corollaries in Proposition LXVI do not give the impression of being organized very systematically. They look more like an accumulation of intuitive insights, all based on the gravitational interaction of three bodies and sometimes argued with a great deal of imagination. The motion of the lunar nodes is seen as closely related to the problem of the tides, as well as the precession of the equinoxes.

G. The Moon in Newton's system of the world (Book III)

Newton tested his grand theory of universal gravitation almost exclusively by studying the motion of the Moon, including its effect on the motion of the Earth as manifested in the tides and the precession of the equinoxes. If one adds up the number of pages in the *Principia* that are devoted to lunar problems, their total probably exceeds the space that is devoted to all the two-body problems, including the discussion of the planetary motions.

Book III, "The System of the World," talks about the Moon more directly rather than citing it only as an example of some general proposition. The rotation of the Earth is discussed, and its flattening at the poles is suggested. The only evidence for this phenomenon at the time was the length of the second pendulum, which was observed to be shorter near the Earth's equator. That leads to the precession of the equinox by 50″/year, of which the Moon accounts for 40″/year and the Sun only 10″/year. The tides are also described in detail with some qualitative explanations.

Then comes the rather audacious claim concerning the motions of the Moon that "*all the inequalities of those motions follow from the principles which we have laid down*." The discussion of proposition LXVI is taken up again with the relevant numerical figures this time. The variation of Tycho Brahe is explained under the assumption of a circular orbit. The perturbation of the Sun produces a closed oval orbit that is centered on the Earth with the long axis in the direction of the quadratures. This ingenious idea would have to wait almost 200 years before being taken up again by G. W. Hill (1877).

The motion of the nodes gets many pages of geometric discussion, including the libration of the inclination and of the nodes. Various annual effects are described and explained qualitatively, and even the figure of the Moon is brought up. But neither the motion of the perigee (with the missing factor 2) nor the evection are mentioned further. In the final analysis, many qualitative explanations concerning the three-body problem are advanced, but only two numbers, the motion of the Moon's perigee and of her node, are obtained. Nothing of substance was added to this record for another 50 years after the first appearance of the *Principia*. Nevertheless, Newton had succeeded in starting a grand unification to the point where hardly anybody could doubt that his somewhat spotty results were only the first signs of a whole new approach to the riddles of nature.

VII. LUNAR THEORY IN THE AGE OF ENLIGHTENMENT

A. Newton on the continent

Although a new age in celestial mechanics as well as in many other branches of the sciences started with Newton, it almost looks as if nobody dared to expand upon his great achievements while he was still alive. The United Kingdom, in particular, needed more than a century to liberate itself from his awesome and fearful presence. On the continent, however, there was a full flowering of his ideas, starting in the late 1730s, fifty years after the *Principia* was published. By the end of the century the marriage of physics with mathematics had been consummated and was producing many healthy offspring.

The French Academy of Sciences decided to test Newton's prediction that the Earth has a flattened shape which is shorter along its axis of rotation by a fraction of a percent. A first expedition under the leadership of Charles-Marie de la Condamine left France in the spring of 1735 to measure an arc of 3 degrees near Quito (Ecuador), from whence it returned successfully in 1744 after incredible adventures and hardships (Condamine, 1751). A second, more high-powered group headed by Moreau de Maupertuis left in the spring of 1737 for Tornea in Lapland, in the north of Sweden, and returned in the summer of 1738 after a job done too well [see Maupertuis, 1756a and 1756b]. Their measure of one degree in the far north yielded some 700 meters more than near Paris, too much by a factor of 2 (see Svanberg, 1835).

But Newton was vindicated against some earlier measurements by the Cassinis, father and son, who had compared the length of one degree in the north and the south of France. Voltaire (1738) had published a very popular and competent ''Elements of Newton's Philosophy,'' and now had a good time greeting the heros returning from Lapland: ''You have flattened the Earth and the Cassinis'' (Terral, 1992). He also encouraged his friend, Gabrielle-Emilie de Breteuil, Marquise du Chatelet, to translate the *Principia* into French: but her work appeared only in 1756, seven years after her death in childbirth. Meanwhile, two French scientist-priests, Thomas Le Seur and Francois Jacquier (1739–1742), had written an extensive commentary on the *Principia* in which many of the obscure passages are more fully explained; they included the three memoirs on the tides by Daniel Bernoulli, Colin McLaurin, and Leonard Euler that had received a prize from the French Academy in 1740.

B. The challenge to the law of universal gravitation

Even the theory of the Moon's motion around the Earth had its moment of drama in this atmosphere of scientific excitement. It involved the three most talented and productive mathematical physicists of the time. In 1736, while a member of the St. Petersburg Academy, Leonard Euler had published the first textbook on Mechanics, in which a plethora of problems were solved for the first time with the help of calculus. Alexis Clairaut had entered the French Academy of Science at the age of 16, had participated in the expedition to Lapland with Maupertuis, and had helped the Marquise du Chatelet in her translation of Newton. Jean Le Rond d'Alembert (1743) was the author of the first treatise on Dynamics, in which Newton's laws were established on the basis of general principles relating to the nature of space and time. All three of them decided, simultaneously but independently, to put lunar theory on a firmer base.

They all submitted their different versions during the summer of 1747 to the Secretary of the French Academy, but found out only during the winter what the others had to say. Euler (1746) had published *New Astronomical Tables for the Motions of the Sun and the Moon* the year before, telling the reader only how to use them, but without explaining how he had computed them. Now he was competing for another prize from the French Academy (which he won as he did 10 others), this one to explain some of the irregularities in the motions of Saturn. Jupiter and Saturn are in a 2/5 resonance, and Euler's treatment showed that he had a lot to say about the Moon as well. Clairaut pointed out that, as a member of the Academy, he like d'Alembert is not allowed to compete for the prize.

All of them wrestled with the problem of the motion of the lunar perigee. They were unanimous in claiming that Newton's law of universal gravitation with the inverse-square-of-the-distance dependence does not account for the observed value, which is larger by a factor of 2, as Newton had already found. In addition, Clairaut (1747) now proclaimed as a great discovery that the distance dependence of the universal gravitation had to be modified for short distances by adding a term in $1/r^4$ (see the Ph.D. thesis of Craig Waff, 1976). He was immediately taken to task by Georges-Louis Leclerc, Comte de Buffon (1747), his famous colleague from the section of natural history, who was unwilling to believe that an important principle of physics could end up leading to a fundamental force with a complicated mathematical form. Was that the last time a representative of the life sciences entered into a lively debate with a theoretical physicist concerning the general principles of the physical sciences?

Clairaut (1752) went back to work a little harder; he pushed his approximations to higher order in the crucial parameter $m=n'/n$, and found the required correction. He deposited the relevant paper at the French Academy in January 1749 while announcing his results without explanation, and submitted his work to the Russian Academy in St. Petersburg for a prize in lunar theory (Clairaut, 1752). Euler was appointed a referee and joined to his report a voluminous treatise of his own, which was eventually published as a separate book, usually referred to as Euler's first lunar theory (1753). Meanwhile, d'Alembert (1749) published a treatise about *Researches on the precession of the equinoxes and the nutation of the Earth's axis*. Together with Diderot he edited the famous *Encyclopédie*, which appeared from 1751 to 1766 in 17 large volumes of text plus 11 volumes of etchings. He also went back to the lunar problem and gave an algebraic, rather than numerical, calculation for the correct value of the motion of the perigee.

Reading all this work nowadays is considerably easier than dealing with Newton, but the authors were still finding their way through the new language of analysis rather than elementary geometry. They did not always find the shortest connections, so much so that Clairaut ended up with Hebrew symbols because the Latin and Greek alphabets were too short. Luckily, their work has been analyzed more recently; first, in the *Historical Essay on the Problem of Three Bodies* published in 1817 by Alfred Gautier; second, by Felix Tisserand in his classic four-volume *Treatise on Celestial Mechanics* from 1889 to 1896, whose third volume is entirely devoted to the theory of the Moon's motion; third, in *An Introductory Treatise on the Lunar Theory* by Ernest W. Brown of 1896.

C. The equations of motion for the Moon-Earth-Sun system

The complete equations for the three-body system Moon-Earth-Sun will now be written down in the most straightforward manner. Since the inertial mass of each of these three bodies always cancels out the gravitational mass, it is simpler to speak directly about the accelerations rather than the forces. We use the nomenclature of Fig. 4, and the masses are again designated by M (Moon), E (Earth), and S (Sun). The accelerations are listed as follows:

$$-\frac{GE}{r^3}\vec{x}-\frac{GS}{\rho^3}\vec{\xi} \text{ of the Moon in } M, \tag{25}$$

$$+\frac{GM}{r^3}\vec{x}-\frac{GS}{r'^3}\vec{x}' \text{ of the Earth in } E, \tag{26}$$

$$+\frac{GE}{r'^3}\vec{x}'+\frac{GM}{\rho^3}\vec{\xi} \text{ of the Sun in } S. \tag{27}$$

The main coordinates for the three-body system are the vector $\vec{x}$ from the Earth to the Moon and the vector $\vec{X}$ from the Sun to the center of mass Γ. Therefore we find that

$$\vec{x}'=\vec{X}-\frac{M}{E+M}\vec{x}, \quad \vec{\xi}=\vec{X}+\frac{E}{E+M}\vec{x}. \tag{28}$$

It takes a little manipulation to end up with the equations of motion,

$$\frac{d^2\vec{x}}{dt^2}=\frac{E+M}{EM}\,\mathrm{grad}_x\left(\frac{GEM}{r}+\frac{GSM}{\rho}+\frac{GSE}{r'}\right). \tag{29}$$

for the Moon with respect to the Earth, and

$$\frac{d^2\vec{X}}{dt^2}=\frac{M+E+S}{S(E+M)}\,\mathrm{grad}_X\left(\frac{GSM}{\rho}+\frac{GSE}{r'}\right), \tag{30}$$

for the center of mass Γ with respect to the Sun. The distances $\rho=|\vec{\xi}|$ and $r'=|\vec{x}'|$ are to be replaced by Eqs. (28). These equations are exact and will form the basis for all our further work.

The vector $\vec{X}$ is about 400 times longer than $\vec{x}$, so that it is natural to expand the denominators in Eqs. (29) and (30). The gravitational potential becomes an expansion in Legendre polynomials P_j, where each term can be interpreted as arising from a multipole. The choice of coordinates ensures that the dipole terms cancel out, leaving the quadrupole term as the lowest-order perturbation to the direct interaction. Thus the right-hand side of Eq. (29) becomes, leaving out the grad_x,

$$\frac{G(E+M)}{r}+\frac{GS}{R}\sum_{j=2}^{\infty}\frac{(-)^jE^{j-1}+M^{j-1}}{(E+M)^{j-1}}\frac{r^j}{R^j}P_j(\cos\omega), \tag{31}$$

where $\cos\omega=(\vec{x},\vec{X})/rR$. The right-hand side of Eq. (30) becomes, leaving out again the grad_X,

$$\frac{G(S+E+M)}{R}\left[1+\frac{EM}{(E+M)^2}\times\sum_{j=2}^{\infty}\frac{(-)^jE^{j-1}+M^{j-1}}{(E+M)^{j-1}}\frac{r^j}{R^j}P_j(\cos\omega)\right]. \tag{32}$$

The trouble in lunar theory arises already in the quadrupole term $j=2$.

The quadrupole term in Eq. (32) is smaller than the monopole term by a factor $(1/400)^2$ from the relative distances r/R, and another factor 1/80 from the relative masses M/E; it will be completely ignored. The solution to Eq. (30) is, therefore, the simple Kepler motion for the center of mass Γ around the Sun. The motion of the vector $\vec{X}$ will henceforth define the plane of the ecliptic, where it moves according to Eqs. (7)–(13). The distance r in (7) will be called R, and the remaining symbols will be decorated with a prime, i.e., a' instead of a, ε' instead of ε, and so on. Kepler's third law in the form of Eq. (15) follows immediately.

The approximate size of the quadrupole term in Eq. (31) can be estimated with the help of the Kepler's third law (15) exactly as in Eq. (22). Its quotient by the monopole term in Eq. (31) is $\sim m^2=(n'/n)^2$, as Newton knew very well. The equation of motion (29) spells out

"The Main Problem of Lunar Theory," that is to find the motion of the Moon relative to the Earth when the center of mass for Earth and Moon is assumed to move on a fixed Kepler ellipse around the Sun, and assuming that the masses of Moon, Earth, and Sun are concentrated in their centers of mass.

D. The analytical approach to lunar theory by Clairaut

Clairaut and many celestial mechanicians after him, including Laplace, chooses the true longitude ϕ rather than the time t as the independent variable. This preference is natural when there are no good clocks available. Similarly, the variable $1/r$ rather than the radial coordinate r represents the lunar parallax (after multiplication with the equatorial radius of the Earth). Clairaut then manipulates the equation for the radial motion into the form

$$\frac{d^2s}{d\phi^2}+s=\Omega, \quad \text{with } s=\frac{F^2}{G_0(E+M)r}-1, \tag{33}$$

where Ω is a somewhat messy expression for the solar perturbation. The parameter F in the definition of s is a constant of motion that comes from the integration of the angular motion.

The motion of the perigee is taken into account from the very start, by inserting

$$r=\frac{k}{1+\varepsilon\cos\mu\phi}. \tag{34}$$

The distance k and the rate $\mu<1$ will somehow emerge from solving Eq. (33). Ω is now expanded as a trigonometric series of ϕ, since the time is everywhere expressed as a function of ϕ with the help of a formal solution for the equation of the angular motion. The constant F turns out to be the average angular momentum of the Moon, with $F=nk^2$.

The perturbation Ω acts like a feedback mechanism for the harmonic oscillator on the left-hand side of Eq. (33) and leads to a resonance of finite amplitude for all frequencies other than 1 that appear on the right-hand side of Eq. (33). Any possible excitation at the frequency 1, however, such as would be caused by the pure Kepler motion, will cause a shift in frequency from 1 to μ rather than an amplitude that goes to ∞.

The terms of order 0 in the perturbation expansion of s cancel one another, provided

$$n^2k^3=G_0(E+M)\left(1-\frac{m^2}{2}\right), \quad \mu^2=1-\frac{3}{2}m^2. \tag{35}$$

The first condition indicates that the rotating ellipse is shrunk compared to the unperturbed Kepler ellipse, which is to be expected because of the additional repulsion due to the Sun. The second condition confirms Newton's result (23) that $\mu-1\cong-3m^2/4$.

E. The evection and the variation

A first-order approximation yields the correction to the rotating Kepler ellipse (34),

$$\frac{k}{r}=1+\varepsilon\cos\mu\phi+\delta\cos(2-2m-\mu)\phi$$
$$+\beta\cos 2(1-m)\phi+\alpha\cos(2-2m+\mu)\phi, \tag{36}$$

where

$$\delta\cong\frac{15}{8}\varepsilon m, \quad \beta\cong m^2, \quad \alpha\cong-\frac{5}{8}\varepsilon m^2. \tag{36$'$}$$

The lunar longitude ϕ as a function of time becomes

$$\phi=nt+2\varepsilon\sin\mu nt+2\delta\sin(2n-2n'-\mu n)t$$
$$+\left(\beta+\frac{3m^2}{8}\right)\sin 2(n-n')t$$
$$+\frac{2\alpha}{3}\sin(2n-2n'+\mu n)t, \tag{37}$$

to the lowest order with respect to m. One recognizes the anomaly, which has the amplitude 2ε, the evection comes with a factor $2\delta=15\varepsilon m/4$, and the variation has the amplitude $11m^2/8$.

High-precision data are customarily expressed in seconds of arc, but we shall stick with the round figures in minutes of arc because they are more easily remembered. Therefore $2\varepsilon\cong375'$ and $m=1/13.3679\cong3/40$ with the help of Eq. (36′) lead to $(\beta+3m^2/8)=26'30''$ and $2\delta=52'36''$, instead of $40'$ and $75'$ for the observed variation and evection. Thus the first-order corrections yield only two-thirds of the observed values.

F. Accounting for the motion of the perigee

The complete first-order expression (36) for the inverse radius k/r as a function of ϕ is now inserted into the radial equation (33) to get the corrections of second order. In trying to cancel out all the terms proportional to $\cos\mu\phi$, one now obtains

$$\mu^2=1-\frac{3}{2}m^2-\frac{225}{16}m^3. \tag{38}$$

The second-order correction to μ is, therefore, $225m^3/32$, compared to the first-order result $3m^3/4$ of Newton. With $225/32\cong7$ and $m\cong3/40$, the second term constitutes 7/10 of the first.

After all this work, Clairaut and d'Alembert were able to account for 85% of the motion of the lunar perigee, compared to Newton's 50%. We have to remind ourselves that the Greeks knew the correct value empirically to better than four decimals. Nevertheless, this work of the two French academicians convinced everybody that Newton's universal gravitation should be sufficient to explain all the motions of the Moon around the Earth. It also suggested that a perfect fit with the observations would be hard to accomplish and could be far down the road.

G. The annual equation and the parallactic inequality

The algebraic expansion (36) provides a tool with which the individual terms in the lunar motion can be derived separately. Brahe's annual equation (see Sec. V.F) was found to have a coefficient $-13'$, whereas the observed value was only $-11'8''$. The negative sign indicates that the Moon falls behind in spring and catches up in fall.

A correction of similar origin arises from the octupole term in the expansion (31) for the perturbation of the Sun. This correction can be singled out because its coefficient has a factor a/a'. The period is the synodic month; such a motion was first noticed by Tobias Mayer (see next section). The calculation yields $73''$ for the amplitude of this "parallactic inequality" in lowest order, whereas the complete value is $125''$. Again, we are short by almost a factor of 2.

If the theory for this inequality could be improved, the ratio a/a' could be obtained from the lunar orbit and, therefore, the solar parallax from the well-known ratio Earth-radius/a. In this manner Tobias Mayer found $8.6''$, which differs insignificantly from the modern value of $8.8''$. A better value for the parallactic inequality, however, is not easy to tease out of the observations. The astronomical unit a', the fundamental measure of length in the universe, is better found by other observations, such as the transits of Venus over the disc of the Sun.

Such transits occur only every 120 years, and then they happen in pairs eight years apart, such as 1761 and 1769 (see Woolf, 1959). The French Academy organized a large enterprise in 1761 which failed, however, due to bad luck and the great war between France and England that spread over the whole globe. (This important event was known as the French-Indian wars in America and led to the ouster of the French from Canada.) The second opportunity in 1769 was more carefully planned and profited from some lucky breaks, such as Captain Cook's discovering Tahiti just in time, and the great Euler himself observing both contacts from a station near St. Petersburg. But even after discarding the more doubtful observations, the final results still varied between 8.5″ and 8.9″ (Newcomb, 1891).

H. The computation of lunar tables

The simple results of the preceding sections demonstrate that the theory had to be greatly refined before complete agreement with the observations could be achieved. The lunar motions were represented as trigonometric series that involved the combination of four angles. The coefficients in these expansions depended on a very small number of parameters, but all through the 18th century the theory was not good enough to carry out the required computations.

Good predictions for the lunar position in the sky were necessary as a help in navigating across the oceans, since mechanical clocks did not run reliably for several weeks or even months without interruption. The existence of the trigonometric series became an accepted result of the general theory. But the values of the coefficients were obtained from fitting the observations, rather than working out any algebraic formulas in terms of the few parameters.

Thousands of lunar positions were provided by the British Astronomers Royal, starting with Flamsteed, followed by the all-round genius Halley, then Bradley, the discoverer of the aberration of light and the nutation of the Earth's axis, and finally Maskelyne, who tested all kinds of new clocks against celestial observations. The Continental astronomers also made many observations in the sky, but they did not accumulate the long runs of measurements with the same instruments under similar conditions. A complete review of all the available data concerning the motion of the Moon was given by Newcomb (1878, 1912).

Although Euler (1746) had already published some tables, and Clairaut (1754) as well as d'Alembert (1756) followed a few years later, it was Tobias Mayer, professor of astronomy at the University of Goettingen, who set the new standards. The claim that his table of 1752 fitted the observations within at most 1′, was reluctantly confirmed by his colleagues. After his death, his theory was published in 1767, and his improved table appeared in 1770 with a preface by Maskelyne. Meanwhile his widow was awarded 3000 pounds by the British Parliament in 1763 to recognize her husband's ability to "Discover the Longitude at Sea," while Euler got 300 pounds for helping Tobias Mayer. Very entertaining accounts of the competition between the Moon and the mechanical clock have been published recently by Andrewes (1993) and Sobel (1995).

Mayer had required 14 linear combinations of angles in the expansion of the Moon's longitude and identified 8 more of them that were too small for the precision of his table. Charles Mason published *Lunar Tables in Longitude and Latitude According to the Newtonian Laws of Gravity* in 1787, using all 20 combinations. (In 1763, upon the request of the Astronomer Royal, Nathaniel Bliss, he had gone with Jeremiah Dixon to the American colonies in order to survey the boundaries between Pennsylvania, Maryland, Delaware, and Virginia; a fictional account of this adventure by Thomas Pynchon appeared in 1997 under the title *Mason & Dixon*.) There followed the Austrian Bürg in 1806, who listed 28 combinations in his table, although there were really 40 terms in longitude because both the parallactic inequality and the variation depended on the mean elongation $D=\lambda-\lambda'$. The last in this roster is the French academician Burckhardt in 1812, who used 36 combinations of angles.

The comparison of individual tables grew more complicated, and the search for the explanation of any discrepancies became more scientific. Methods of least squares were used for judging the quality of the last two tables and for finding the best values for the mean motions and the so-called epochs, i.e., the value of the relevant angles at some arbitrary time such as midnight before January 1, 1801. The root of the mean-square deviation went down to 6″ of arc, although exceptionally an individual deviation might reach almost 60″. The coefficients in the trigonometric series for the longitude of the Moon were given to the tenth of a second of arc. The observations were still ahead of the theory.

I. The grand synthesis of Laplace

Pierre Simon Laplace (1749–1827) is a well-known figure in the French scientific pantheon, not necessarily for his many scientific achievements, but rather for his philosophical approach to them and for his influence on present-day institutions in France. Nevertheless, with his monumental four-volume work *Traité de Mécanique Céleste*, divided into ten books and published from 1799 to 1805 (the fifth volume, covering history and some general physics, followed 20 years later), Laplace validated Newton's claim that all of astronomy in the solar system can be reduced to the three laws of motion and universal gravitation with the inverse square of the distance.

The whole work is characterized by its precise as well as concise language and its systematic buildup, starting from general principles and ending with the fine details of comparison with the observations. An abbreviated version with the title *Mechanism of the Heavens* was published in 1831 by Mary Sommerville, from a Scottish middle-class family. It contains a discussion of lunar theory far beyond what is offered in this review. She became well known and well established, but she could not be elected to the Royal Society of London. Instead the Fellows decided to have her marble portrait made by

Francis Chantrey and to have it stand in Burlington House, their headquarters.[1]

English-speaking readers have been blessed with a unique translation of the first four volumes of Laplace's treatise. A self-made businessman from Boston, Nathaniel Bowditch (1773–1838), not only translated but also added numerous comments to explain with infinite patience exactly what Laplace was doing. The resulting four tomes (Bowditch, 1829, 1832, 1834, and 1839) contain about two and a half times as many pages as the original and constitute a prime document of science in the early United States. Its author also wrote a voluminous *New American Practical Navigator* that went through dozens of editions, and he received many honors for his efforts from academic institutions the world over.

The developments in *Mécanique Céleste* do not follow some general method, nor do they pretend to provide a complete survey of the whole field, but they cover many of the outstanding problems: the shapes and the rotation of the bodies in the solar system, including the tides in the ocean and in the atmosphere, as well as the rings of Saturn, the motion of the individual planets and their satellites with all the mutual perturbations and resonances, and finally a discussion of the comets, but ignoring the recent discovery of the asteroids. Laplace naturally prefers to present much of the successful work that he had done in the preceding 30 years. He tries to push every topic to a perfect agreement between observation and theory, and he succeeded in convincing the world that such a goal could be attained. He backed up this claim with sophisticated arguments from his theory of probability.

J. Laplace's lunar theory

Book VII in the third volume treats the "Theory of the Moon." After 12 pages of general introduction, there follow 123 pages of hard analysis. Bowditch had to expand them to 331 pages in order to accommodate all his explanatory remarks. The spirit is still the same as in the work of Clairaut and d'Alembert, but the motion in latitude is taken into account from the start, and many more terms are included in the trigonometric series. The basic equations are a fairly straightforward generalization of Eq. (33), and demonstrate once more the contrast with the various modern approaches.

Starting from polar coordinates (r,θ,ϕ) for the Moon with respect to the ecliptic, the equations of motion are written in terms of

$$u=\frac{1}{r\cos\theta},\quad \sigma=\tan\theta,\quad \phi, \tag{39}$$

where the true longitude ϕ serves as the independent variable. The differential equations for u and σ look almost exactly like Eq. (33). Laplace used the "variation of the constants" of Lagrange in book VI, the first part of the third volume, which is devoted to the perturbations in the planetary orbits. But he preferred the more primitive approach of Clairaut and d'Alembert for the Moon.

The starting point for Laplace's lunar theory is a modified Kepler motion in three dimensions. In addition to the motion of the perigee μ, there is now a motion of the node ν already in the lowest approximation,

$$u=\frac{\sqrt{1+\sigma^2}+\varepsilon_0\cos\mu(\phi-\omega)}{k},\quad \sigma=\gamma_0\sin\nu(\phi-\Theta). \tag{40}$$

The eccentricity ε_0 and the inclination γ_0 differ somewhat from the earlier definitions for ε and γ. Together with the angles ω and Θ they are the initial conditions for the integration of the equations of motion.

The lunar motion now looks like a dynamic problem of two oscillators with feedback mechanisms. The frequency has to shift for both the radial motion and the motion in latitude because otherwise there would be an infinite resonance with the motion in longitude and latitude. The lowest approximation for μ is again given by Eq. (35), while the lowest approximation for ν is obtained from Eq. (40) and yields Newton's result (24), $\nu-1\cong+3m^2/4$.

The further development of the lunar motion at a right angle to the ecliptic follows the same pattern as the earlier calculations in Sec. VII.E. In complete analogy to the evection, the first-order corrected formula for the latitude becomes

$$\sigma=\gamma_0\sin\nu(\phi-\Theta)-\frac{3m}{8}\gamma_0\sin[\nu(\phi-\Theta)-2(1-m)\phi)], \tag{41}$$

where we have kept only the lowest power of the $m=n'/n$. In spite of its similar origin, the evection term in Eq. (36) is five times larger with a coefficient 15/8 in α rather than the libration with 3/8 in Eq. (41).

Since γ_0 corresponds to $5°11=311'$, and $3m/8\cong(3/8)(3/40)=9/320$, this change in latitude amounts to about $9'$, which is the observed value. Equation (41) can also be interpreted as a motion of the nodes by finding the longitude ϕ when the latitude σ vanishes. Thus Θ is found to vary as $(3m/8)\sin 2D$, i.e., with an amplitude of 1°37, in good agreement with the observations of Brahe (cf. Sec. V.F). Again we find that the motion perpendicular to the ecliptic comes out quite well in the lowest significant approximation, in contrast to the motion in the ecliptic.

A great deal of care is necessary if the method of Laplace is to be carried to fourth or even fifth order. The details in *Mécanique Céleste* are not always easy to follow because the numerical value for the most important (and best known) parameter is used, namely, the ratio of the mean motions $m=n'/n\cong3/40$, or any function of it. Also, there are subtle arguments of what we call "renormalization" because the starting parameters in the theory are not always identical with the observed ones.

[1]I happen to own the copy of *Mechanism of the Heavens* that Sommerville dedicated to the sculptor Chantrey.

The work of Laplace had profound philosophical implications for the sciences because he was able to achieve what we would call today a ''grand unification'' for everything that was observed in the solar system. Since the main problem in lunar theory requires very few external parameters plus some initial conditions, it affords an extremely stringent test for both the underlying physics and the mathematical methods. The agreement with the observations of Bradley and Maskelyne averages around one arcsecond. Nothing of this kind had ever been achieved before!

Laplace also investigated the effect of the planets on the Moon's motion and found that it is mostly indirect, i.e., through the motion of the Sun around the center of mass of the solar system. Halley had discovered that the Moon's motion was accelerating slowly, and Laplace found the cause in the variation of the Earth's eccentricity. Tobias Mayer had found an inequality in longitude that depended only on the position of the node, and Laplace now attributed it to the flattened figure of the Earth, whose value he determines thereby as 1/305.

The great endeavor of Laplace has left a deep mark on our view of nature, but his methods for solving the equations of celestial mechanics are no longer used. The many technical inventions that led to the success of this project have either been abandoned as impractical or become part of the common background in this field. The guiding spirit of this great enterprise has influenced many of our beliefs, but the huge bulk that supported the whole structure is mostly forgotten.

VIII. THE SYSTEMATIC DEVELOPMENT OF LUNAR THEORY

A. The triumph of celestial mechanics

The 19th century brought many new problems in celestial mechanics because of three major discoveries: (i) the major planet Uranus in 1781, (ii) four large asteroids from 1801 to 1807, and (iii) the last major planet, Neptune, on the basis of the observed perturbations on Uranus and the calculations of Urbain Leverrier and John Couch Adams in 1846.

The second half of the 19th century brought some clarification into the many approaches to celestial mechanics, particularly in three respects:

(i) one special approach to mechanics, now associated with the names of Hamilton and Jacobi, seemed to become dominant;
(ii) the lunar problem served as inspiration for Hill to invent a completely new foundation;
(iii) the many schemes for constructing approximate solutions were finally examined from a purely mathematical perspective by Poincaré.

At the same time, the comparison of the theory with ever better observations was improved, and the computations for the ephemerides were made more accurate and practical.

Section IX deals with (i) and (iii), because much of Poincaré's work in celestial mechanics is directly based on classical mechanics in the Hamilton-Jacobi version. By contrast, Hill's work throws a totally new light on the motion of the Moon and will be discussed in Sec. X. These developments, however, began with Lagrange, who seems to be the first mathematical physicist trying to find general methods while solving special problems. Together with his much younger colleague Poisson, he worked out a general formalism that allowed a clearer insight into the lunar motion.

B. The variation of the constants

Euler's first theory (1753) for the motion of the Moon was only mentioned in passing in Sec. VII.B. As in so many other instances, Euler was far ahead of his time, but he followed up on his new approaches only to the extent required to solve the problem at hand. Some of his most interesting results are found in the Appendix.

Rather than solving the equations of motion in any of the various coordinate systems, Euler's simple idea was to express the rate of change with time, for any of the osculating elements in Sec. V.B. The time derivatives of the parameters a, ε, γ, ℓ_0, g, h for the Moon are given directly in terms of the perturbation by the Sun. Thanks to his virtuoso skills in geometry and analysis he was able to carry out this program.

Giuseppe Lodovico Lagrangia (1736–1813) spent his first thirty years in his native Torino, where he started most of his scientific work. He founded a new journal together with some like-minded friends to publish his prodigious output. In 1766 he was called to Berlin as the successor of Euler, who was moving back to St. Petersburg. Thanks to the eminence of Euler and Lagrange, Frederick the Great of Prussia succeeded in putting his Royal Academy on the map. Some 147 years later, the same academy offered its senior position to the 33-year-old Albert Einstein, a Swiss citizen (like Euler) who had spent some of his youth in Italy, to complete the analogy.

Although he got an early start on his seminal ideas, it took Lagrange more than 50 years of hard work to cast them into the simple shape that we learn about in classical and quantum mechanics. This long process began in 1774 with his research on the secular variation of the nodes and inclinations of planetary orbits. The adjective secular refers to a slow change in the speed of the mean motion. Applying this research to lunar theory, he introduced the angular momentum vector (in modern nomenclature) of the Moon with respect to the Earth, and calculated its time rate of change when the lunar orbit is assumed to be circular in lowest approximation.

The fruit of this approach came two years later when Lagrange (1776) derived his famous theorem on the stability of the solar system. The semi-major axis a' for some particular planet like the Earth is given by the total energy of its motion around the Sun, $-G_0SE/2a'$. The only reason for the change of a' is the interference of the other planets. Again we insert in lowest approximation the simple Kepler motions for each planet.

The secular variation of the semi-major axes had been calculated earlier by Euler, who got rather large values. Laplace then carried Euler's computation one step further and showed that there was a subtle compensation of the two lowest orders. With Lagrange's simple expression for da'/dt it became almost obvious that there was no secular, i.e., constant as opposed to periodic, term. In Jacobi's words, the proof of Lagrange was accomplished with one stroke of his pen (Jacobi, 1866).

In 1808 Poisson showed the absence of secular terms in second order provided there is no resonance between any two planets, i.e., there is no linear relation with simple integers between the mean motions of any two planets. Poincaré (1899) pointed out in Chapter XXVI that Poisson excluded only terms like Bt from the change of the semi-major axis, but admitted terms like $At\,\sin(\alpha t+\beta)$. Stability to second order only implies the return to the original values, but allows arbitrarily large excursions in between. It was finally shown that secular terms could no longer be avoided in third order (see Tisserand's discussion in Chapter XXV of Volume I).

From 1781 to 1784 Lagrange generalized his method by including what we now call the Runge-Lenz vector [Eq. (14)]. The change of these quantities due to perturbations leads to a change in a, ε, γ and the angles h, g, ℓ_0. In this roundabout way, Lagrange calculated the time rate of change for the Kepler parameters a, γ, etc. Lagrange made a major effort to check whether his scheme gave the observed values for the planetary motions, to the point of critically evaluating various observing instruments. Obviously, the separation between observation and theory was not yet as wide as it is today.

During the same years Lagrange wrote his magnum opus, *Analytical Mechanics*, the founding document for modern theoretical physics. The central idea of Lagrange's approach was expressed in the "avertissement" (i.e., preface), where he says "*There are no figures in this treatise. The methods that I propose require neither constructions nor mechanical reasoning, but only algebraic operations that are bound to a regular and uniform procedure. People who like analysis will see with pleasure that mechanics has become a part of it, and they will be grateful to me for having expanded its range.*"

C. The Lagrange brackets

In 1787 Lagrange joined the Royal Academy of Sciences in Paris where he was treated with the utmost respect to the point of being offered the doubtful privilege of an apartment in the Louvre. In the same year the *Analytical Mechanics* was published, and Lagrange spent most of his remaining 25 years in writing mathematical monographs, typically on the solution of algebraic equations and on the theory of functions. But he also worked hard on a second and expanded edition of the *Analytical Mechanics*, of which the second volume was not quite completed when he died. In 1808 he made a major breakthrough by finding a general method for solving problems in mechanics.

Lagrange does not speak of the momentum, but always of the velocity; even today mathematicians usually do not make that distinction, although it is already quite clearly stated on the very first page of Newton's *Principia*. In gravitational problems the distinction is somewhat artificial, however, because the mass of any celestial body always drops out of its equation of motion since the inertial mass equals the gravitational mass.

Lagrange's reasoning can be best explained with the help of the Moon's motion around the Earth. If there is no perturbation, the complete solution of Kepler's problem would be a set of functions,

$$x(a,\varepsilon,\gamma,\ell_0,g,h;t),\quad y(a,\ldots,h;t),\quad z(a,\ldots,h;t),$$
$$\dot{x}(a,\varepsilon,\gamma,\ell_0,g,h;t),\quad \dot{y}(a,\ldots,h;t),\quad \dot{z}(a,\ldots,h,;t). \tag{42}$$

The coordinates and velocities depend on time in two ways: through the explicit occurrence of t in the Kepler motion as represented in Eq. (42), and through the change of the parameters a, ε, γ, ℓ_0, g, h because of the perturbation.

Since the coordinates (x,y,z) are explicitly known as functions of the parameters $a,\ldots,h$ and t, the perturbing potential $W(x,y,z,t)$ is now viewed also as a function of these parameters. By a sequence of simple manipulations, Lagrange manages to write the equations of motion in the following form:

$$\{a,a\}\frac{da}{dt}+\{a,\varepsilon\}\frac{d\varepsilon}{dt}+\cdots+\{a,h\}\frac{dh}{dt}=-\frac{\partial W}{\partial a},$$
$$\{\varepsilon,a\}\frac{da}{dt}+\{\varepsilon,\varepsilon\}\frac{d\varepsilon}{dt}+\cdots+\{\varepsilon,h\}\frac{dh}{dt}=-\frac{\partial W}{\partial \varepsilon}, \tag{43}$$

and four more equations for γ, ℓ_0, g, h, where the Lagrange bracket $\{\varepsilon,a\}$ is defined by

$$\{\varepsilon,a\}=m_0\left(\frac{\partial(x,\dot{x})}{\partial(\varepsilon,a)}+\frac{\partial(y,\dot{y})}{\partial(\varepsilon,a)}+\frac{\partial(z,\dot{z})}{\partial(\varepsilon,a)}\right), \tag{44}$$

with

$$\frac{\partial(x,\dot{x})}{\partial(\varepsilon,a)}=\frac{\partial x}{\partial \varepsilon}\frac{\partial \dot{x}}{\partial a}-\frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial \varepsilon}.$$

The 6×6 matrix of the Lagrange brackets is regular, because it is basically the square of the Jacobian matrix for the functions (42). Calculating the individual brackets, however, requires all of Lagrange's computing skills. Most importantly, he shows that their partial derivative $\partial/\partial t$ vanishes. Each bracket is a combination of the six parameters a, ε, γ, ℓ_0, g, h; but t does not occur explicitly in them, even though it is present in the functions (42).

The 6×6 matrix of the Lagrange brackets turns out to be so simple that the linear equations (43) can be inverted to yield

$$\frac{da}{dt}=-\frac{2}{na}\frac{\partial W}{\partial \ell_0}, \tag{45}$$

$$\frac{d\ell_0}{dt}=\frac{2}{na}\frac{\partial W}{\partial a}+\frac{1-\varepsilon^2}{na^2\varepsilon}\frac{\partial W}{\partial \varepsilon}, \tag{46}$$

$$\frac{d\varepsilon}{dt}=\frac{\sqrt{1-\varepsilon^2}}{na^2\varepsilon}\frac{\partial W}{\partial g}-\frac{1-\varepsilon^2}{na^2\varepsilon}\frac{\partial W}{\partial \ell_0}, \tag{47}$$

$$\frac{dg}{dt}=-\frac{\sqrt{1-\varepsilon^2}}{na^2\varepsilon}\frac{\partial W}{\partial \varepsilon}+\frac{\cos\gamma}{na^2\sqrt{1-\varepsilon^2}\sin\gamma}\frac{\partial W}{\partial \gamma}, \tag{48}$$

$$\frac{d\gamma}{dt}=\frac{1}{na^2\sqrt{1-\varepsilon^2}\sin\gamma}\frac{\partial W}{\partial h}-\frac{\cos\gamma}{na^2\sqrt{1-\varepsilon^2}\sin\gamma}\frac{\partial W}{\partial g}, \tag{49}$$

$$\frac{dh}{dt}=-\frac{1}{na^2\sqrt{1-\varepsilon^2}\sin\gamma}\frac{\partial W}{\partial \gamma}. \tag{50}$$

The reader should notice that the parameters a, ε, γ change with time only through the partial derivatives of the perturbation W with respect to the angles ℓ_0, g, h. These derivatives are necessarily periodic with time, so that a, ε, γ are subject only to periodic changes in first order. Lagrange's theorem on the invariance of the semi-major axes (the stability of the solar system) is a special case of this conclusion.

For many bodies in the solar system the eccentricity ε and the inclination γ are small. In these cases the singularities on the right-hand sides of Eqs. (46) through (50) can be avoided if the parameter pairs (ε,g) and (γ,h) are replaced by $(\varepsilon\cos g,\varepsilon\sin g)$ and $(\gamma\cos h,\gamma\sin h)$. Poincaré will use this idea when he discusses the convergence of power-series expansions in these parameters. The main task now becomes to write the perturbing potential $W(x,y,z,t)$ as a function of parameters such as our $(a,\varepsilon,\gamma,\ell_0,g,h)$.

D. The Poisson brackets

The 27-year-old Poisson was inspired by the paper that the 72-year-old Lagrange read before the French Academy of Sciences. Within two months Poisson had found a significant shortcut by starting with the opposite of Eq. (42): the constants of motion, say our usual collection for the Kepler problem, are given explicitly in terms of the Cartesian coordinates (x,y,z) and their velocities $(\dot{x},\dot{y},\dot{z})$.

It follows then with relative ease that

$$\frac{da}{dt}=-[a,a]\frac{\partial W}{\partial a}-[a,\varepsilon]\frac{\partial W}{\partial \varepsilon}-\cdots-[a,h]\frac{\partial W}{\partial h},$$

$$\frac{d\varepsilon}{dt}=-[\varepsilon,a]\frac{\partial W}{\partial a}-[\varepsilon,\varepsilon]\frac{\partial W}{\partial \varepsilon}-\cdots-[\varepsilon,h]\frac{\partial W}{\partial h}, \tag{51}$$

and similar equations for the remaining four parameters. The Poisson bracket $[A,B]$ between any two functions $A(x,y,z,\dot{x},\dot{y},\dot{z})$ and $B(x,y,z,\dot{x},\dot{y},\dot{z})$ is defined as

$$[A,B]=\frac{1}{m_0}\left(\frac{\partial(A,B)}{\partial(x,\dot{x})}+\frac{\partial(A,B)}{\partial(y,\dot{y})}+\frac{\partial(A,B)}{\partial(z,\dot{z})}\right). \tag{52}$$

The 6×6 matrix of the Poisson brackets is the inverse of the 6×6 matrix of the Lagrange brackets. Indeed, the Poisson brackets for the Kepler problem are exactly the coefficients in the equations (45)–(50) of Lagrange.

Lagrange did not live long enough to see Poisson brackets steal the show from his own brainchild. He had tried to grasp the most basic elements of mechanics, and he almost succeeded, with the help of his method of virtual displacements and of his general equations of motion in terms of the kinetic and potential energies. In the process, classical mechanics became an ever more abstract branch of science. In particular, lunar theory after Lagrange turned more and more into a competition among different computational schemes.

E. The perturbing function

In order to take advantage of our new equations of motion, the perturbing potential $W(x,y,z,t)$ has to be rewritten. The Cartesian coordinates $(x,y,z,)$ have to be expressed as functions of the Kepler parameters $(a,\ldots,h)$ and time t as in Eq. (42).

This messy calculation is carried out in exemplary fashion by Delaunay (1860, 1867; see Sec. IX.B). The parameters $\varepsilon\cong1/18$, $\gamma=\sin\gamma/2\cong1/22$, and $\varepsilon'\cong1/60$ are considered as first-order quantities, whereas $a/a'\cong1/400$ is treated as second order. All terms in W up to eighth order are listed. Moreover, terms whose arguments contain ℓ' but not ℓ are carried to ninth order, while terms containing neither ℓ nor ℓ' are listed to tenth order. These terms have a slow periodic variation, so that their integration introduces small denominators.

The result of this computation is given explicitly on pages 33–54 of the first volume. It forms the basis for the remaining 1750 pages, which will be discussed in the next chapter. Here, we shall copy some of the very lowest-order terms,

$$W=-\frac{G_0Sa^2}{a'^3}\left(\frac{1}{4}+\frac{3}{8}\varepsilon^2-\frac{3}{8}\gamma^2+\frac{3}{8}\varepsilon'^2-\frac{1}{2}\varepsilon\cos\ell+\frac{3}{4}\cos2(h+g+\ell-\Phi)+\frac{15}{8}\varepsilon^2\cos2(h+g-\Phi)+\frac{3}{8}\gamma^2\cos2(h-\Phi)\right), \tag{53}$$

where we have set the Sun's mean longitude $\ell'+g'+h'=\Phi$.

The general term contains a cosine of the argument in the form

$$j_1(h+g+\ell-h'-g'-\ell')+j_2\ell+j_3\ell'+j_4(g+\ell). \tag{54}$$

Its coefficient is a polynomial, where the individual term is a rational number that gets multiplied with

$$\frac{G_0Sa^2}{a'^3}(a/a')^{k_1}\varepsilon^{k_2}\varepsilon'^{k_3}\sin^{k_4}(\gamma/2). \quad (55)$$

The coefficient j_1 and all the exponents k are $\geqslant 0$. The following relations hold: j_1 and k_1 are simultaneously even or odd; j_4 and k_4 are both even; k_2, k_3, k_4 are larger, respectively, than $|j_2|$, $|j_3|$, $|j_4|$ by an even number $\geqslant 0$.

Any physical understanding of the results is possible only if the individual terms in the expansion of W can be tracked down to the original expression for the solar perturbation. Looking at the 22 pages of Delaunay's expansion, it is clear that only the very lowest terms, such as Eq. (53), can be identified and interpreted. Unfortunately, it lies in the very nature of modern theoretical physics that such obvious reductions are no longer feasible. Our gain in precision does not necessarily come with a better understanding.

F. Simple derivation of earlier results

Some of the principal perturbations in the lunar trajectory were calculated in the preceding section just as they were explained for the first time by the French mathematicians in the 18th century. These same results can now be obtained much faster with the help of Lagrange's method. The expansion (53) has to be inserted into the equations of motion (45)–(50), which then have to be integrated. Since the mean motion n is expressed through Kepler's third law [Eq. (15)], i.e., $n=\sqrt{G_0(E+M)/a^3}$, the mean anomaly at epoch has to be redefined,

$$\ell = nt+\ell_0=\int n\,dt+\ell_1, \quad (56)$$

in order to avoid terms of the type $t\sin\ell$; ℓ_1 serves as the new Kepler parameter.

From the first line in Eq. (53) we get immediately

$$\frac{d\ell_1}{dt}=-\frac{7}{4}\frac{n'^2}{n},\quad \frac{dg}{dt}=\frac{3}{2}\frac{n'^2}{n},\quad \frac{dh}{dt}=-\frac{3}{4}\frac{n'^2}{n}, \quad (57)$$

in agreement with Newton's results (23) and (24). The correction to the longitude $\ell+g+h$, i.e., $\delta(\ell_1+g+h)$, is found to decrease at a rate n'^2/n. It is directly related to a renormalization of the average distance of the Moon from the Earth, and to correction (35) of Kepler's third law.

The differential equations (45)–(50) are solved in lowest approximation by integrating the right-hand sides under the assumption that a, ε, γ have constant values, while ℓ, g, h increase at a fixed rate with time. The first term in the second line of Eq. (53) yields Tycho Brahe's variation. The second term in the second line of Eq. (53) yields Ptolemy's evection, and the third line in (83) gives the lowest-order correction (41) to the motion in latitude.

Lagrange's equations of motion (45)–(50) can also be used to find the corrections in the Moon's motion for the nonspherical nature of the Earth. The perturbing potential depends on the declination of the Moon with respect to the Earth's equator and on the geophysical parameters in the Earth's gravitational field, as first established by Clairaut (1743). The answers were finally figured out by Laplace, who found the correction $\delta\beta=-8''.382\sin(\ell+g+h)$ in latitude and $\delta\lambda=7''.624\sin h$ in longitude. The first had been observed by Tobias Mayer without interpretation, and the second was discovered subsequently by Bürg and Burckhardt. This problem plays a central role in our times for the trajectories of artificial satellites around the Earth (see end of Sec. XI.E).

G. Again the perigee and the node

The improved values of Clairaut and d'Alembert for the motions of the perigee and of the node can be obtained rather easily from Lagrange's equations (47)–(50) in the following ingenious manner. Only the terms proportional to ε^2 and γ^2 in the perturbation (53) are taken into account. If the longitude of the perigee is called $\omega=g+h$, one finds that

$$\frac{d\varepsilon}{dt}=-\frac{3m^2}{4}5\varepsilon\sin 2(\lambda'-\omega),$$
$$\frac{d\omega}{dt}=\frac{3m^2}{4}(1+5\cos 2(\lambda'-\omega)), \quad (58)$$
$$\frac{d\gamma}{dt}=-\frac{3m^2}{4}\gamma\sin 2(\lambda'-h),$$
$$\frac{dh}{dt}=-\frac{3m^2}{4}(1-\cos 2(\lambda'-h)), \quad (59)$$

where $\lambda'=\ell'+g'+h'$ is the mean longitude of the Sun. We shall assume that $\lambda'=n't+\lambda_0'$ where n' is known. Some small coupling between these two sets of equations has been neglected. The two pairs of equations were found by Puiseux in 1864 to be completely integrable, with solutions that use only elementary functions.

Rather than displaying the full solutions, it is sufficient to say that h, the longitude of the node, is found to increase on the average at the rate $n'(1-\sqrt{1+3m/2})$, whereas the longitude of the perigee $\omega=g+h$ increases at the rate $n'(1-\sqrt{(1+3m)(1-9m/2)})$ where $m=n'/n$. When these expressions are expanded in powers of m, one finds

$$\dot h=-n'\left(\frac{3m}{4}-\frac{9m^2}{32}+\cdots\right),$$
$$\dot g+\dot h=+n'\left(\frac{3m}{4}+\frac{225m^2}{32}+\cdots\right). \quad (60)$$

Remarkably, these are the correct values for the first two terms of the expansions in powers of the relevant parameter m [cf. Eq. (33)].

In his discussion of Newton's lunar theory, Tisserand (Vol. III, p. 44) mentions the collection of Newtonian manuscripts that the Count of Portsmouth left to the University of Cambridge. A committee, including Stokes and Adams, examined the papers and found important results in only three areas: lunar theory, atmospheric refraction, and the form of a solid of least resistance. Newton had investigated the motion of the perigee for an orbit of small eccentricity, and had formulated two lemmata as if for a fourth edition of the *Principia*. They are equivalent to the second equation (58), except that the coefficient 5 becomes 11/2 for reasons that are not well understood. But even so, there is a significant improvement for the motion of the perigee over Newton's earlier value. (See the comments of Chandrasekhar, 1995.)

IX. THE CANONICAL FORMALISM

A. The inspiration of Hamilton and Jacobi

William Rowan Hamilton (1805–1865) is a romantic figure in the best tradition of the early 19th century, supremely gifted and at times deeply unhappy. Yet his enormous talents were widely appreciated, and he received many honors, such as being at the head of the first 14 foreign associates in the U.S. National Academy of Sciences at its founding in 1863. His unique achievement in physics was to recognize the deep analogy between optics and mechanics (see Hamilton's *Mathematical Papers*, 1931 and 1940).

The analog of Fermat's principle in optics is the variational principle of Euler and Maupertuis in mechanics (Euler, 1744). The relevant characteristic function is the integral of the momentum at constant energy $E=T+V$ along a trajectory from the initial point (x_0,y_0,z_0) to the endpoint (x,y,z). But in 1834, Hamilton considered the variation for the more general integral of $L=T-V$ over the trajectory from the initial spacetime (x_0,y_0,z_0,t_0) to the final (x,y,z,t). He also wrote down the first-order partial differential equation for these characteristic functions. In 1837 he had an extended correspondence with Lubbock on the motion of the Moon, although his interests remained mostly in optics.

Carl Gustav Jacob Jacobi (1804–1851) was Hamilton's equal in mathematical talent, but he seems to have been better organized and more effective. In mechanics, he started in 1837 where Hamilton left off and created a whole body of theory that has become the foundation of the modern approach to classical mechanics. It is well explained in his famous *Vorlesungen über Dynamik*, which he held 1842–43 in Königsberg, now Kaliningrad. They were published by Clebsch only in 1866 together with some further well-written, but so far unpublished, notes on the theory of perturbation.

Jacobi's main emphasis was on obtaining the characteristic function from its first-order partial differential equation, the classical analog of Schrödinger's equation. This first-order partial differential equation is solved explicitly whenever there is a sufficient number of constants of the motion, e.g., in the Kepler problem, and the solution yields the orbit directly in terms of the relevant parameters. The procedure is explained in all the standard textbooks on classical mechanics with various degrees of abstraction.

The main purpose of present-day attention to Hamilton-Jacobi theory is the connection between classical and quantum mechanics. This possibility was not foreseen in the first half of the 19th century. The theory found relatively few applications and was probably responsible for some illusions that still persist. Indeed Jacobi and all his successors (with the important exception of Poincaré) seem to imply that most dynamic systems are integrable, i.e., they have as many integrals of motion as degrees of freedom, if you are only smart enough to find them.

Although there is some merit in this idea for the double-planet Earth-Moon moving around the massive Sun, the construction of the required constants of motion cannot be accomplished by some miraculous application of Hamilton-Jacobi theory. Although canonical transformations go back to Hamilton and Jacobi, their practical use in the service of perturbation theory cannot be traced back to them.

B. Action-angle variables

The first large-scale application of canonical transformations was worked out by Charles-Eugène Delaunay (1816–1872), a prominent engineer, mathematician, astronomer, professor, and academician. In 1846 he proposed a "New Method for the Determination of the Moon's Motion" on which he worked for the following 20 years all by himself while teaching and writing various textbooks and monographs. The detailed record of these heroic labors is contained in two monumental tomes, each over 900 pages, published in 1860 and 1867.

There is no reason to believe that Delaunay knew anything about the work of Hamilton and Jacobi, and yet there is no better example of the use of canonical transformations in perturbation theory. Delaunay wrote with great precision and clarity, with the necessary detail to check on his computations. And indeed, that has been done with the tools of both the 19th and the 20th century.

In a major change, Delaunay describes the pure Kepler motion of the Earth-Moon system in terms of an angular momentum μL instead of the usual semi-major axis a. The energy becomes

$$-\frac{G_0EM}{2a}=-\mu\,\frac{(G_0EM)^2}{2\mu^2L^2},\quad \text{where}\quad \mu=\frac{EM}{E+M}. \tag{61}$$

With a few obvious modifications, one recognizes the energy levels of the hydrogen atom.

Two minor changes involve the angular momenta, μG and μH instead of the eccentricity ε and the inclination γ,

$$L=\sqrt{G_0(E+M)a},\quad G=L\sqrt{1-\varepsilon^2},\quad H=G\cos\gamma. \tag{62}$$

The division of the three angular momenta by the reduced mass μ simplifies some of the formulas, but we shall continue to speak about the angular momenta L,G,H. They are paired with the mean anomaly ℓ, the distance g from the ascendant node to the perigee, and the longitude h of the ascendant node from the reference direction Q in the ecliptic. In modern language, we have the "actions" L,G,H and the angles ℓ, g,h; they are commonly called the Delaunay variables.

The equations of motion now take the standard form for a perturbed system,

$$\frac{d\ell}{dt}=\frac{\partial\Omega}{\partial L},\quad \frac{dg}{dt}=\frac{\partial\Omega}{\partial G},\quad \frac{dh}{dt}=\frac{\partial\Omega}{\partial H},$$
$$\frac{dL}{dt}=-\frac{\partial\Omega}{\partial\ell},\quad \frac{dG}{dt}=-\frac{\partial\Omega}{\partial g},\quad \frac{dH}{dt}=-\frac{\partial\Omega}{\partial h}, \tag{63}$$

where Ω is now the kinetic plus potential energy divided by μ,

$$\Omega=-\frac{(G_0(E+M))^2}{2L^2}+W(x,y,z,t). \tag{64}$$

Notice that the derivatives in Eq. (63) no longer require the special precaution of Eq. (56) in order to prevent the appearance of terms where the time t multiplies a trigonometric function of the angles.

The expansion (53) of W in powers of ε, γ, ε', and a/a' can be used again; indeed, it was worked out by Delaunay in exactly this form (see Sec. VIII.E). But the Kepler parameters a, ε, γ are now expressed in terms of L,G,H,

$$a=\frac{L^2}{G_0(E+M)},\quad \varepsilon=\sqrt{1-\frac{G^2}{L^2}},$$
$$\sin\frac{\gamma}{2}=\sqrt{\frac{1}{2}-\frac{H}{2G}}. \tag{65}$$

Obviously, Delaunay has made a compromise between two viewpoints: on the one hand, the Kepler parameters are very helpful in understanding a particular term in Ω and estimating its magnitude; on the other hand, the computations are made easier by the simplicity of Eqs. (63) compared to Lagrange's equations (45)–(50).

For the sake of symmetry in the nomenclature, a fourth degree of freedom is introduced to take care of the Sun's motion. Its action variable is called K, and its angle is $k=\ell'$. A fourth pair of equations will appear in Eq. (63), and the total potential Ω will include an additive term $n'K$ to make sure that the angle ℓ' increases at the constant rate n'. This fourth degree of freedom does not interact with the other three, but the energy of the Earth-Moon system depends on time through ℓ'. This situation is described nowadays as due to 3 plus 1/2 degree of freedom.

Delaunay uses the expansion of the perturbation W in Sec. VIII.E. Each term in this trigonometric expansion is transformed away individually, one after the other, by a procedure that yields all the higher-order corrections. The factors (55) determine which of the new terms to retain and how far to carry the procedure. Although the whole process is very systematic, it is still beset by many details that require tremendous attention as well as almost infinite patience.

C. Generating functions

The Swedish mathematician von Zeipel is generally credited with having developed the use of generating functions in performing canonical transformations around 1916, although there were precursors. In his Ph.D. thesis of 1868, Tisserand discusses Delaunay's results in this manner (see Tisserand Vol. III, Chap. 11); Poincaré also uses generating functions in his *Nouvelles Méthodes* as if they were common property.

There is a set of old actions and angles, K, L, G, H, and k, ℓ, g, h, and a set of new actions and angles, K', L', G', H', and k', ℓ', g', h'. *The generating function F depends on the new actions and the old angles*; the transformation is given by the formulas

$$K=\frac{\partial F}{\partial k},\quad L=\frac{\partial F}{\partial\ell},\quad G=\frac{\partial F}{\partial g},\quad H=\frac{\partial F}{\partial h},$$
$$k'=\frac{\partial F}{\partial K'},\quad \ell'=\frac{\partial F}{\partial L'},\quad g'=\frac{\partial F}{\partial G'},\quad h'=\frac{\partial F}{\partial H'}. \tag{66}$$

Naturally, the generating function has to be chosen to advance the solution of the problem at hand.

In contrast to Delaunay we shall now assume that the resulting canonical transformation does not differ by much from the identity. This allows us to transform at the same time several terms of the type $A\cos\theta$ where A is a function of the old actions such as Eq. (55), and θ is some linear combination with integer coefficients of the old angles such as Eq. (54). Thus we start from the Hamiltonian in the form

$$\Omega=B(K,L,G,H)+A_1\cos\theta_1+A_2\cos\theta_2+\cdots. \tag{67}$$

where $A_1,A_2,\ldots$ are small compared to B. Then we construct a generating function as a sum of several pieces,

$$F=K'k+L'\ell+G'g+H'h+F_1+F_2+\cdots. \tag{68}$$

where F_1 is designed to take care of $A_1\cos\theta_1$, and F_2 of $A_2\cos\theta_2$, etc.

Since the time does not occur explicitly in the old Hamiltonian Ω, the new Hamiltonian Ω' is obtained from the old one simply by replacing in Ω the old actions and angles by their expression in terms of the new ones. Equations (66) are applied to Eq. (68) and inserted into Eq. (67). The choice of the functions F_1, F_2, and so on is dictated by the requirement that the new Hamiltonian in terms of the new actions and angles have no more terms such as $A_1\cos\theta_1$, $A_2\cos\theta_2$, etc.

The condition for the vanishing of the term $A_1\cos\theta_1$ in Ω then yields

$$F_1=-\frac{A_1(L',G',H')\sin\theta_1}{(i_{10}\omega_0+i_{11}\omega_1+i_{12}\omega_2+i_{13}\omega_3)}, \tag{69}$$

and similar expressions for F_2, and so on. The integers $i_{10},i_{11},i_{12},i_{13}$ are the same as in $\theta_1=i_{10}k+i_{11}\ell+i_{12}g+i_{13}h$. The ω's are the frequencies of the undisturbed system and are given by the standard formulas

$$\omega_0=\frac{\partial B}{\partial K}=n', \quad \omega_1=\frac{\partial B}{\partial L}, \quad \omega_2=\frac{\partial B}{\partial G}, \quad \omega_3=\frac{\partial B}{\partial H}. \tag{70}$$

This is where the small denominators raise their ugly heads. With Von Zeipel's method, one can obtain the generating function for the lowest-order corrections simply by looking at the Hamiltonian Ω. In order to proceed further, one must expand the transformation formulas from the old action angles to the new ones, to higher powers in A_1, A_2, etc. The results of these expansions have to be inserted into Eq. (67), and this last step will produce additional terms that are independent of the angles, as well as terms with new combinations of angles θ. But these new terms in the Hamiltonian Ω' are expected to be smaller than those that were eliminated by the generating function (68). Finally the lunar coordinates have to be expressed in terms of the new action-angle variables, a big job.

D. The canonical formalism in lunar theory

Delaunay first carries out a set of 57 transformations and eliminates all terms in the original Hamiltonian of order lower than four. The detailed record of this major task makes up the first volume; the main effort goes into finding the new terms of higher order in the Hamiltonian that are produced by the transformations of the lower-order terms. Since the remaining 440 transformations no longer interfere with one another, they are equivalent to a single transformation with a generating function to lowest order. The numerical accuracy of the Moon's longitude is not quite satisfactory at this point, however, and some of the earlier operations have to be improved. Therefore, 8 new canonical transformations are added, for a grand total of 505.

Since the motions of the angles are given by the derivatives (70), the term $B(K,L,G,H)$ in the final Hamiltonian is the most important result. Deprit, Henrard, and Rom of the Boeing Scientific Laboratories in Seattle reported in 1970 that Delaunay's final expression for B has to be corrected by subtracting the single term $(5/8)n'^2a^2m\,\sin^2(\gamma/2)\varepsilon'^2$, where the lowest term in perturbation (53) is $(1/4)n'^2a^2$. The effect of this correction on the rate of change with time for the angles ℓ_0, g, h in the original Kepler problem can be obtained directly from Eqs. (46), (48), and (50). For the motion of the node, the correction is

$$\Delta\,\frac{dh}{dt}=\frac{5}{16}\,\frac{n'^2}{n}\,m\varepsilon'^2, \tag{71}$$

whereas the correction vanishes for the motion of the perigee, $d(g+h)/dt$. This correction amounts to about 10^{-5} of Newton's result $-3n'^2/4n$.

The same authors published in 1971 a more detailed comparison of Delaunay's results with their own computations, which were carried to higher order and will be discussed in Sec. X. Delaunay's final expression for the Moon's longitude is a trigonometric series in the angles (54) of 460 terms covering 53 pages, while his expression for the latitude includes the angles of 423 terms covering an additional 52 pages. The coefficient for each term is a polynomial in $m=n'/n$, ε, $\sin(\gamma/2)$, ε', a/a', all with rational numbers. 49 corrections in these polynomials were necessary for the longitude and 45 for the latitude. An earlier suggestion by Andoyer (1901) was confirmed, namely that most of the terms of order 8 and 9 are erroneous, altogether a somewhat disappointing comparison.

Delaunay's work is a benchmark for what one human individual is able to accomplish without the help of computing machines. His results can be looked up in a good library, since they were published in 1860 and 1867. The more extensive work that came out of the computers in the 1970s and 1980s, however, is not easily available because both the programming and the technical prowess of the machines have changed so rapidly. Whoever wants to find out what has already been done, beyond a qualitative account, is almost forced to do the whole job all over again with the help of whatever means are available at the time.

Delaunay provided important insights into the convergence of series expansions in celestial mechanics. This convergence is poor in the ratio $m=n'/n$, which is known to very high accuracy. Although Delaunay tried to go to ninth order in m, some of his results are still insufficient, e.g., his expression for the motion of the perigee is still in error by 10^{-4}. On the other hand, his theory is analytic in all the variables and yet comparable in accuracy to the best of his time, i.e., Hansen's, who had used numerical values for all the variables from the very start.

E. The critique of Poincaré

At the end of the 19th century, classical mechanics took a decisive turn away from the happy optimism of its earlier practitioners. Physicists are finally waking up at the end of the 20th century to an unpleasant reality that the mathematicians, followed by engineers and astronomers, have known for a long time: even simple dynamic systems, with only two degrees of freedom and conserved energy, such as the double pendulum, have very complicated motions as a rule. The elementary examples of the textbooks like Kepler's motion for an isolated planet are not typical at all of most of the realistic systems in nature.

This capital discovery can be safely attributed to Henri Poincaré (1854–1912), whose first series of scientific papers is concerned with the qualitative behavior of the solutions of ordinary differential equations. This early work led him into celestial mechanics, in particular, into a study of periodic solutions of the full three-body problem and its various special cases. This activity received a strong stimulus when King Oscar II of Sweden announced a prize for the best scientific paper to prove (or disprove) the stability of the solar system. Poincaré was eventually declared the winner although he was unable to give a definitive answer to the main question, and his long paper was published in 1890 in *Acta Mathematica* (see Barrow-Green, 1996).

Then he followed up with what is considered his masterpiece, a three-volume work entitled *The New Methods of Celestial Mechanics*, which appeared in 1892, 1893, and 1899. Its English translation has just been published, a century later. This is not the place to give even a brief account of this monumental work, except to say that the Moon plays a special role as the first example for which a periodic orbit was chosen as the starting point for a new approach in mechanics, an idea that was first proposed in 1877 by Hill to explain the Moon's motion (see next section). Working through the more than 1200 pages of mathematical argument is not for the faint-hearted and can be frustrating because Poincaré insists on using a fairly abstract language, in spite of the evident inspiration from physics, astronomy, and celestial mechanics.

Three volumes of *Lectures on Celestial Mechanics* were published in 1905, 1907, and 1910. They were neither intended as a rehash of the *New Methods of Celestial Mechanics* nor as a competition to the classic *Treatise of Celestial Mechanics* by Felix Tisserand, in four volumes (1889, 1891, 1894, 1896), of which the third is entirely devoted to the Moon. The standard methods are subjected to a mathematical scrutiny in order to establish their legitimacy, in particular with respect to the convergence of the approximation schemes. The first volume treats the motion of the planets and is dominated by the methods of Hamilton and Jacobi. The first part of the second volume discusses the purely technical problem of expanding the Hamiltonian for the perturbation calculation, while the second part examines the mathematical justification for the lunar theory of Hill and Brown. The rather hefty third volume treats the many kind of tides, in oceans and rivers, in the Earth's crust, and in the stars, both in theory and in observation.

F. The expansion of the lunar motion in the parameter *m*

In 1908 Poincaré published a 40-page paper in the *Bulletin Astronomique* with the title "On the small divisors in the theory of the Moon." Its length is due to the many different cases that have to be taken up in the argument, but it is rather straightforward and its notation stays close to the special conditions of the Moon's motion. Rather than discussing the convergence of the solution that comes out of perturbation theory, the main issue is more drastic. The question to be answered will be phrased carefully in order to get a clear reply.

The lunar problem in the form (63), with the time replaced by the angle k as in Sec. IX.B, has four pairs of action-angle variables plus the small parameters $m=n'/n$, ε, γ, ε', and $\alpha=a'/a$. The two pairs (K,k) and (L,ℓ) are left as they are, but the pairs (G,g) and (H,h) are replaced by two others in order to reflect the fact that both the eccentricity ε and the inclination γ are small. In agreement with Eq. (62) and the end of Sec. VIII.C, we shall use

$$\rho=\sqrt{2(L-G)},\quad \sigma=\sqrt{2(G-H)} \tag{72}$$

to form the new pairs $(\rho,g+h)$ and (σ,h). Obviously, ρ is of the order $\varepsilon\sqrt{L}$, and σ is of the order $\gamma\sqrt{L}$. With the further change in variables to (ξ_1,η_1) and (ξ_2,η_2), where

$$(\xi_1,\eta_1)=\rho(\cos(g+h),\ \sin(g+h)),$$
$$(\xi_2,\eta_2)=\sigma(\cos h,\ \sin h), \tag{73}$$

we have again two pairs of conjugate variables.

These *Poincaré variables* can be used directly in the perturbation W to arrange the many terms in decreasing order of importance according as the powers of m, $\rho=\sqrt{\xi_1^2+\eta_1^2}$, $\sigma=\sqrt{\xi_2^2+\eta_2^2}$, ε', and α. Each term has a "characteristic,"

$$m^{\mu_1}\rho^{\mu_2}\sigma^{\mu_3}\varepsilon'^{\mu_4}\alpha^{\mu_5}, \tag{74}$$

whose exponents μ give an approximate idea of its relative size. Generating functions are used exactly as described in Sec. IX.B.

Equations (60), however, show that the denominator in Eq. (69) may bring about a division by various powers of m, up to the third in the special case when the motion of the node $\dot{h}/n$ and the motion of the perigee $(\dot{g}+\dot{h})/n$ enter with the same multiple. Poincaré calls denominators of this type "analytically very small," in contrast to denominators where they enter in the ratio 2 to 1, which he calls "numerically very small" because of the empirical values of the lunar periods in Sec. V.C. Therefore certain unavoidable canonical transformations will lower the exponent μ_1 in the characteristic.

Poincaré goes through a careful and rather lengthy examination of all the many cases that might arise and how they change the characteristic in the individual terms of the solution. The interference of the small parameters besides m complicate the discussion, but ultimately the presence of three degrees of freedom leads to disaster: If Delaunay's expansion is pushed far enough, terms are found where the exponent of m is negative!

This conclusion defeats the whole purpose of the theory, even though it happens only when the expansion is driven quite far, much further than in Delaunay's work. Paradoxically, his incomplete solution might still provide an excellent numerical accuracy. But this state of affairs is worse than in an ordinary asymptotic expansion, like that for the Bessel functions where the coefficients increase so fast, e.g., like factorials, that the positive powers of the expansion parameter are defeated. In the lunar theory negative powers in the most crucial parameter cannot be avoided at all!

X. EXPANSION AROUND A PERIODIC ORBIT

A. George William Hill (1838–1914)

Hill's career reflects life in the United States at the end of the 19th century, with one glaring exception. He never adjusted to the amenities and strains of regular academic and scientific life, even after his work was widely appreciated and he received many honors. When his Collected Works were published in 1905, Poincaré wrote the 12-page introduction (in French), where he says "*This reserve, I was going to say this savagery, has been a happy circumstance for science, because it has allowed him to complete his ingenious and patient researches*."

Hill's father moved to the countryside a few years after George's birth in New York City, to start farming in West Nyack, 30 miles up the Hudson river. Since his mathematical ability had been noticed, he was sent to Rutgers College in New Jersey, where he was lucky to find a first-class teacher who made him study the classical works from the 18th and early 19th century. In 1861, he joined the staff of the scientists working in Cambridge, Massachussets, on the *American Ephemeris and Nautical Almanac*.

After Simon Newcomb became *Superintendent of the Nautical Almanac* in 1877, Hill started working on the theory of Jupiter and Saturn. His results, with the title *New Theory of Jupiter and Saturn* form volume III of the Collected Papers, and occupy a hefty tome of more than 500 pages. It was a cornerstone in Newcomb's great project of revising all the data for the orbits in the solar system. (Physicists are aware of this enterprise only because it definitely established the missing 42″ in the centennial precession of Mercury's perihelion, which were the best data for the confirmation of Einstein's theory of general relativity; see Pauli 1921 and 1958.) After a ten-year stay in Washington, D.C., Hill retired in 1892 to his beloved farm in West Nyack.

Hill's great contribution to all of mechanics (including its quantum version), and in particular to its celestial branch, is contained in the 1877 paper *On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and the Moon*. It was followed 1878 by a more detailed version *Researches in Lunar Theory*. In the Collected Works, these papers appear as numbers 29 and 32, among some 80 others that cover a wide range of topics.

Lunar theory at that time came in several different versions besides Delaunay's. The theory of Lubbock and de Pontécoulant appeared in the 1830's, and was a clever mixture of elements from earlier efforts that could be systematically expanded in powers of the Kepler parameters. Hansen (1838, with additions in 1862–1864), on the other hand, started from equations close to Lagrange's variation of the constants (45)–(50). But he did not develop a systematic perturbation theory, although he did carry his computations to very high precision. His final expression for the lunar coordinates became the accepted standard for the remainder of the 19th century, until it was replaced in 1923 by Ernest W. Brown's extension of Hill's work in all the national ephemerides.

B. Rotating rectangular coordinates

Hill's inspiration must be connected with Euler's Second Theory, which was published (1772) in St. Petersburg under Euler's direction with the help of fellow academicians, J.-A. Euler (his son), Krafft, and Lexell. It introduces two new ideas that turned out to be critical: The Moon is described by Cartesian coordinates with respect to the ecliptic, and this reference system turns around the Earth with the Moon's mean motion n in longitude. This trick eliminates the linear term in time from the longitude, just as there is none in latitude and sine parallax. Euler also is the first to order the terms in the expansion of the lunar motion according to their characteristics like Eqs. (55) or (74).

But Hill makes a crucial modification by letting the coordinates turn with respect to the ecliptic at the rate of the mean motion of the Sun n' rather than of the Moon. The acceleration of the Moon in this rotating system is now changed by $2n'(-dy/dt,+dx/dt,0)$, the negative of the Coriolis force, and by $n'^2(-x,-y,0)$, the negative of the centrifugal force.

The series (31) is again the basis for the expansion of the solar perturbation, but the Legendre polynomials are handled differently. Instead of treating r^j as a factor in front of P_j, Hill puts it inside, so that r^jP_j becomes a homogeneous polynomial of degree j in the lunar coordinates. (In most solid-state applications of atomic theory, the Legendre polynomials of low order are always written in this explicit manner.)

Hill's purpose is to take into account the terms that depend only on the ratio $m=n'/n$. Therefore he sets $R=a'$, and all the corrections in powers of ε' are left out in lowest approximation. Moreover, he stops with the solar quadrupole, because the higher multipole terms have additional powers in $\alpha=a'/a$. With $n'^2=G_0S/a'^3$, the primary perturbation now reads simply

$$V=-n'^2\left(\frac{3}{2}x^2-\frac{1}{2}(x^2+y^2+z^2)\right). \tag{75}$$

The work of Newton, then of Clairaut and d'Alembert, and eventually of many others, including particularly Delaunay, had shown that the expansion in powers of m was the main culprit for making lunar theory so difficult and unsatisfactory. Poincaré's results (at the end of the last section) were still in the future. Hill was the first to treat the difficulties with the motion of the perigee completely, and separately from all the other complications in the motion of the Moon.

C. Hill's variational orbit

With the quadrupole potential (75) added to the standard gravitational attraction between Earth and Moon, there is now a well-defined mechanical problem of motion in three dimensions that has to be solved completely, at least for sufficiently small values of m. Before one writes down its equations explicitly, the length scale is normalized to the semi-major axis a according to Kepler's third law (15). The time variable τ is normalized to the length of the synodic month,

$$\tau=(n-n')(t-t_0). \tag{76}$$

With the complex notation

$$u=x+iy, \quad v=x-iy, \quad r^2=x^2+y^2+z^2=uv+z^2,$$

$$\zeta=\exp(i\tau), \quad D=\frac{d}{id\tau}=\zeta\,\frac{d}{d\zeta}, \tag{77}$$

the equations of motion now are

$$D^2u+2m'Du-\frac{u}{r^3}+\frac{3m'^2}{2}(u+v)=0, \tag{78}$$

$$D^2v-2m'Dv-\frac{v}{r^3}+\frac{3m'^2}{2}(u+v)=0, \tag{79}$$

$$D^2z-\frac{z}{r^3}-m'^2z=0, \tag{80}$$

where we have introduced the new parameter $m'=n'/(n-n')=1/$(number of synodic months in one year).

Since our dynamic system does not have any time dependence, there is an integral of motion C which Hill calls the Jacobian integral,

$$\frac{1}{2}(-DuDv+Dz^2)-\frac{1}{r}-\frac{3}{8}m'^2(u+v)^2=C. \tag{81}$$

Normally, one would first of all define the energy of a trajectory by fixing the value of C, but Hill decides to find a solution that is defined by its period right from the start, not by its energy.

His ingenious idea is that the real trajectory of the Moon must be close to a periodic orbit with the correct period. Among all the known correction terms in the accepted description of the Moon's motion, only Tycho Brahe's variation is retained because it has the period of the synodic month. The resulting "variational orbit" lies in the ecliptic and has a slightly oval shape which is centered on the Earth, with the long axis in the direction of the half moons, as Newton had foreseen.

The required periodicity with the synodic month is enforced by postulating a solution in the form

$$u_0(\tau)=\frac{1}{a}(x_0+iy_0)=e^{i\tau}\left(a_0+\sum_{j=1}^{\infty}(a_j\zeta^j+a_{-j}\zeta^{-j})\right), \tag{82}$$

a power series with positive and negative integer exponents. If the orbit is symmetric with respect to the x axis, then $u_0(-\tau)=u_0^*$, and all the coefficients in the series are real. The symmetry with respect to the y axis is equivalent to the point symmetry at the origin, so that $u_0(\tau+\pi)=-u_0(\tau)$, and all the odd-numbered coefficients vanish.

Hill's 1878 paper contains a detailed and complete account of his work in finding the variational orbit in the ecliptic, $z=0$. The problem is nonlinear because of the terms with $1/r^3$ in Eqs. (78) and (79). The coefficients in Eq. (82) are expanded in powers of m' to exponents that guarantee 15-figure accuracy! Poincaré (1907) made Hill's procedure more transparent in his *Lectures on Celestial Mechanics*. With the help of the Jacobian integral, Hill was able to manage his computations in rational arithmetic, and in such a way that he gained a factor m'^4 at each step.

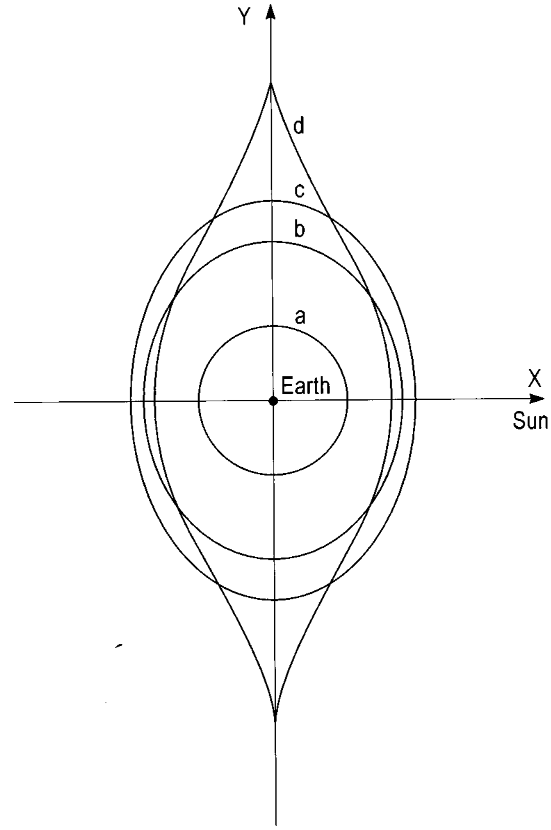

FIG. 5. Hill's diagram of the variational orbit for different lengths of the synodic month: $a=12.369$ lunations per year, like our Moon, $b=4$, $c=3$, $d=1.78265$ lunations leading to a cusp. Poincaré showed that this series continues smoothly, leading to an ever increasing loop around the half moon.

At the end of this long paper, Hill obtains the variational orbit for increasing values of m'. The number of synodic months per year$=1/m'$ comes down from 12.37 through the integers until it reaches the critical value 1.78265 where the oval shape acquires a cusp in the half moons (see Fig. 5). Hill thought the satellite would not reach the half moon position any longer for supercritical values of m'. But Poincaré showed that the sequence of variational orbits continues with the satellites now making a short regressive motion around the half moon in the rotating frame, before taking up again the principal motion in the forward direction.

A recursive computation for the variational orbit was set up by Schmidt (1995) with relative ease, provided the program can handle rational arithmetic. The expansion of u_0 is made in powers of m' where each term is a finite polynomial in ζ and ζ^{-1} whose order does not exceed the power of m'. Each next-higher order follows from the lower-order terms in a standard recursion applied to Eq. (78).

D. The motion of the lunar perigee

The real trajectory of the Moon is now constructed as a small displacement from the variational orbit. To make the distinction, coordinates of the variational orbit will carry the suffix 0, while the displacement in first order carries the suffix 1. The perigee and the node will be treated simultaneously because they lead to almost the same equations. Our presentation will differ somewhat from the classical work of Hill.

The linearized version of Eqs. (78)–(80) is written as

$$(D+m')^2u_1+M(\zeta)u_1+N(\zeta)v_1=0,$$

$$(D-m')^2v_1+M(\zeta)v_1+N^*(\zeta)u_1=0, \tag{83}$$

$$D^2z_1-2M(\zeta)z_1=0. \tag{84}$$

The variational orbit (82) enters these equations through

$$M(\zeta)=\frac{m'^2}{2}+\frac{1}{2r_0^3},\quad N(\zeta)=\frac{3m'^2}{2}+\frac{3u_0^2}{2r_0^5}, \tag{85}$$

where the complex conjugate N^* has v_0 replacing u_0.

Hill's work is concerned with the two equations (83); we shall not discuss their solution in detail, but explain briefly in the next section how to solve the simpler equation (84) in a more direct manner. The differential equations (83) and (84) are linear with coefficients that are periodic functions of the independent variable τ; their general discussion goes back to Floquet (1883). The trivial solution $(Du_0,Dv_0,0)$ is eliminated by considering only the displacement w in the ecliptic that is locally at a right angle to the variational orbit. It satisfies "Hill's equation,"

$$D^2w=\Theta w,\quad \text{with } \Theta=\sum_{-\infty}^{+\infty}\theta_j\zeta^{2j} \text{ and } \theta_{-j}=\theta_j. \tag{86}$$

The periodic function Θ is a rather complicated expression in terms of the variational orbit (82). This oscillator is driven parametrically with the frequency 1, but it responds with the different frequency c_0 that gives the motion of the perigee.

By a daring maneuver, Hill transforms his equation to an infinite set of homogeneous linear equations with a determinant $\Delta(c_0)$ that can be reduced to the simple form

$$\Delta(c_0)=\Delta(0)-\frac{1-\cos c_0\pi}{1-\cos\sqrt{\theta_0}}. \tag{87}$$

The solution of $\Delta(c_0)=0$ is thereby reduced to an expansion of $\Delta(0)$ in powers of m'^4. The motion of the perigee c_0 is obtained to very high accuracy. Poincaré remarks in his preface to Hill's *Collected Papers: "In this work, one is allowed to perceive the germ of most of the progress that Science has made ever since."*

E. The motion of the lunar node

The first application of Hill's method obviously concerns the motion of the lunar node. The luckless Adams, who was a close second to Leverrier in the discovery of the planet Neptune, published a short notice in 1877 after reading Hill's great paper. He had obtained the motion of the node some years before in exactly the same fashion, starting from Eq. (84). The discussion of "Hill's equation" in this case is simpler, and the origin of "Hill's determinant" is easier to understand; but Adams does not get the formula (87) for Δ.

It is worth emphasizing the discoveries that are hidden in this method. First, a periodic orbit is found in some appropriate reference frame that has to be chosen on the basis of physical intuition. Then the dynamic neighborhood of this orbit is examined by linearizing the equations of motion. The result of this approach is always an equation that reads exactly like Hill's equation (86).

The Moon's motion at right angle to the ecliptic is described by a first-order displacement z_1 from the variational orbit,

$$iz_1(\tau)=e^{ig_0\tau}G(\tau)-e^{-ig_0\tau}G^*(\tau), \tag{88}$$

where G is also complex valued, satisfying the periodicity conditions $G(\tau+\pi)=G(\tau)$. We assume the expansion

$$G(\tau)=\sum_{-\infty}^{+\infty}\kappa_j\zeta^{2j}, \tag{89}$$

where the coefficients are real. The function Θ in Hill's equation is now the function $M(\zeta)$ in Eq. (85) from which one gets the coefficients θ_j.

The coefficients in Eq. (89) can be viewed as a vector κ, and Hill's equation is easily reduced to a system of linear equations,

$$\theta(g_0)\kappa=0. \tag{90}$$

The infinite matrix $\theta(g_0)$ has the same structure as the matrix in the preceding section that led to the determinant $\Delta(c_0)$. The motion of the lunar node is obtained by requiring that the determinant of $\theta(g_0)$ vanish. In terms of the angles g and h, we get the more familiar expressions for the mean motion of the perigee and of the node,

$$\dot g+\dot h=\left(1-\frac{c_0}{1+m'}\right)n,\quad \dot h=\left(1-\frac{g_0}{1+m'}\right)n. \tag{91}$$

F. Invariant tori around the periodic orbit

Condition (90) for the coefficients in the displacement z_1 leaves an arbitrary constant to be determined for the motion (88) at right angle to the ecliptic. This parameter plays the same role as an initial condition and is clearly related to the ordinary inclination γ. The term $j=0$ in Eq. (89) by itself describes the essential feature of the motion at right angle to the ecliptic, so that κ_0 can be taken as a measure of the effective inclination γ; it serves as the initial condition in Eq. (84).

In the same manner, the condition $\Delta(c_0)=0$ with Eq. (87) leaves us with the choice of one real number that determines the scale of the displacement u_1 in the ecliptic. The number is equivalent to the effective eccentricity ε of the lunar orbit, and acts like an initial condition for the equations of motion (83).

The multiplicity of solutions for the lunar problem can be given a geometric interpretation. The "flow" of the lunar trajectories is embedded in a phase space of six dimensions, three for the Cartesian coordinates and three for the components of the momentum. This six-dimensional space is naturally reduced to the five-dimensional surface of constant energy by the Jacobian integral (81).

The most general trajectory for the Moon is given in lowest approximation by the variational orbit (82) in the ecliptic, with the displacements (86) and (88). While the variational orbit is covered in one synodic month corresponding to the time variable τ, the anomaly $\ell = c_0 \tau$ runs with the anomalistic month, and the argument of latitude $F = g_0 \tau$ runs with the draconitic month. The result is a three-dimensional torus surrounding the variational orbit, all embedded in a five-dimensional space.

The speeds c_0 and g_0 at which any trajectory winds itself around the variational orbit are independent of the eccentricity ε and inclination γ. They do not vanish as the torus winds itself ever more closely around the variational orbit. The higher-order corrections change nothing in this first-order picture. Hill's theory gives a complete description of the flow in phase space near the real lunar trajectory. It is less than what the ambitious earlier theories were trying to find, since it covers only a relatively small part of the phase space for the original problem (29).

G. Brown's complete lunar ephemeris

Ernest William Brown (1866–1938) came from his native England in 1891 to teach mathematics at Haverford College, where he wrote his much appreciated *Introductory Treatise on the Lunar Theory* (Brown, 1896), and the long series of papers (Brown, 1896–1910) to expand Hill's work into a complete description of the Moon's motion. In 1907 he went to Yale University, mainly because of promised support for the computing and publishing of his lunar tables, and he eventually became the first J. W. Gibbs Professor of Mathematics. The tables finally appeared in 1919 and became the base for the calculations of all the national ephemerides after 1923; see the biography by Hoffleit (1992).

The complex notation (77) is now completed by the definitions,

$$\zeta = e^{i\tau}, \quad \zeta_1 = e^{ic_0\tau + \ell_0}, \quad \zeta_2 = e^{ig_0\tau + F_0},$$
$$\zeta_3 = e^{im'\tau + \ell'_0}, \tag{92}$$

where the new complex variables are associated with the effective eccentricity ε, the effective inclination γ, and the eccentricity ε' of the Earth-Moon's orbit around the Sun. The complete expansion of the lunar coordinates now becomes

$$\begin{pmatrix} \zeta^{-1} u \\ iz \end{pmatrix} = \sum_{pqrs} A_{pqrs} \zeta_1^p \zeta_2^q \zeta_3^r \zeta^s, \tag{93}$$

where even powers of ζ_2 go with $\zeta^{-1}u$, while the odd powers of ζ_2 go with iz.

The coefficients A_{pqrs} are real polynomial series in the variables ε, γ, ε', $\alpha = a/a'$,

$$A_{pqrs} = \sum_{abcd} C_{pqrs}^{abcd} \varepsilon^a \gamma^b \varepsilon'^c \alpha^d. \tag{94}$$

The coefficients C depend on the frequency ratio m' as well as on the mass ratios M/E and $(E+M)/S$ and satisfy the inequalities

$$a \geq |p|, \quad b \geq |q|, \quad c \geq |r|. \tag{95}$$

The expression $\mu = \varepsilon^a \gamma^b \varepsilon'^c \alpha^d$ is called by Brown the *characteristic* of a particular term in Eq. (93). The lowest characteristic that is compatible with a given set of exponents (p,q,r) is its *principal characteristic*.

The successive terms in Eq. (93) can be regarded as higher-order displacements from the variational orbit. Each one can be determined by insertion into Eqs. (78)–(80), which have to be augmented with the additional terms of the solar perturbations that were left out of Eq. (75). The characteristics in Eq. (93) provide a natural ordering in which the variational orbit represents the order 0, while the displacements u_1 and z_1 are the first order; they determine the higher orders.

For each higher order an equation such as (83) and (84) is found, in which the new term in Eq. (93) appears on the left-hand side, whereas the relevant terms of lower order appear on the right-hand side. Therefore the mechanical analog is now an externally driven parametric oscillator; its frequency is determined by the combination of zeta's. Instead of Eq. (90) one gets an inhomogeneous linear equation such as

$$\theta(pc_0 + qg_0 + rm')\,\eta = Q, \tag{96}$$

where g_0 has been replaced by a linear combination with integer coefficients, $pc_0 + qg_0 + rm'$. The vector Q on the right-hand side arises from the right-hand side in the modified equations (83) and (84).

Solving the equations for the lunar problem can now be organized quite systematically according to the characteristics, and it is always clear before beginning the process of computing the right-hand sides in Eq. (96) which characteristics of lower order are able to contribute to the given characteristic μ, and how. The actual solution at each step requires no more than solving an inhomogeneous linear equation with always the same matrix. Even for high-accuracy work, the number of vector components at any one step is no more than 20 because the functions (85) converge like power series in $m'\zeta$.

The matrix θ in Eq. (96) is regular unless, because of Eq. (90), $q=1$ and $p=r=0$ in the argument of θ. Such a thing can happen when two terms of comparatively high characteristic meet on the right-hand side of the equations of motion (84). The system will respond with a shift in frequency δg_0, and the required corrections of the frequency can be worked out.

H. The lunar ephemeris of Brown and Eckert

When Brown started to work on the new lunar theory based on Hill's ideas, he stated his aim very clearly. The accuracy was to be such "*that the coefficients of all periodic terms in longitude, latitude, and parallax shall be included which are greater than 0″.01, and that they shall be correct to this amount. The number of terms required is undoubtedly very great. The calculation of coefficients up to sixth order inclusive with respect to the lunar eccentricity and inclination will be necessary; those of the seventh order may be replaced by their elliptic values*."

Hansen's theory, which was in universal use then, was entirely numerical, while Delaunay's, which was being completed to include the effects of the planets, was entirely algebraic. Brown (1896–1910) made an important compromise: the ratio $m'=n'/(n-n')$ of the mean motions would be replaced by its numerical value from the very start; its value was extremely well known and very unlikely to change due to new observations; any expansion in powers of m' seemed to require the most troublesome mathematics, and the results were generally no guarantee of improvement in the general accuracy. On the other hand, leaving ε, γ, ε', and α as algebraic quantities was part of the whole development; their values were more subject to revision, and keeping them algebraic would facilitate the comparison between different theories.

The resulting *Tables of the Motion of the Moon* (1919) are the last of their kind; the first tables of similar but much simpler design are found in Ptolemy's *Almagest*. Their use requires only looking up with interpolation, and addition of numbers in order to get the coordinates of the Moon in the sky for any instant of time. A lot of ingenuity goes into making the task of the computer, a human being at that time, straightforward and reliable.

In 1932 Comrie proposed the use of mechanized bookkeeping machines for the purposes of celestial mechanics. This project was realized by Wallace J. Eckert, professor of astronomy at Columbia University, who established in 1933 the T.J. Watson Astronomical Computing Bureau with the help of IBM's chairman. During the second World War, Eckert was the director of the U.S. Nautical Almanac Office at the Naval Observatory in Washington, D.C., where he took special pride in designing the first Air Almanac that was produced without human intervention and presumably free of printing errors. After the war, he became the first Ph.D. to be hired by IBM and was asked to run its first research laboratory, to be located at the campus of Columbia University.

Meanwhile it had become clear that, while the claimed accuracy had indeed been achieved in Brown's original papers, the tables did not quite live up to expectations. In the process of making the results usable for the practical calculation of ephemerides, a number of short-cuts had been adopted that reduced their precision. In 1954, Eckert and his collaborators published a careful list of all coefficients in the expansion (93) that Brown had obtained and converted them directly to similar expansions for the lunar longitude, latitude, and parallax. In using this list, however, the computer was expected to multiply each coefficient with its appropriate trigonometric function, and then add up all the products. There are 985 terms from the solar perturbation and 642 additional terms for the planetary perturbations, where all of the latter are smaller than 1 arcsecond. This "Improved Lunar Ephemeris" became the basis for the Apollo program.

Eventually Eckert decided to redo all of Brown's work with the algebra done on one of IBM's all-purpose electronic computers. After his retirement in 1967, he continued his project with the help of just one programmer. All terms in Eqs. (93) and (94) including the sixth order were going to be calculated with an accuracy of 10^{-12}. After Eckert's death in 1971, I accepted the job of seeing the work to its conclusion. A detailed comparison with related work (Gutzwiller, 1979) showed that the expected accuracy had been achieved, but the rigid goal of all sixth-order terms was unrealistic because of the incredible proliferation of minute terms without practical significance.

Dieter Schmidt (1979, 1980a) of the University of Cincinnati proceeded independently to redo Brown's calculations with the help of his own program for the algebraic manipulation of large trigonometric series. With the large increase in the accuracy of the lunar data, it seemed natural to include all the terms larger than 10^{-12}, independent of their order. Since Schmidt's results could be directly compared with the results of Eckert's project, the two works were reported together (Gutzwiller and Schmidt, 1986). The conclusions from this state-of-the-art computation will be discussed in the next section.

XI. LUNAR THEORY IN THE 20TH CENTURY

A. The recalcitrant discrepancies

The last chapter in the third volume of Tisserand's *Treatise on Celestial Mechanics* (1984) deals with the "Present State of Lunar Theory"; it does not cover Brown's Complete Ephemeris and the finishing touches by Eckert, nor the critical evaluation of Poincaré. Nevertheless, Tisserand's conclusions are more farsighted than he might have expected: he acknowledges some fundamental difficulties and hopes for a major discovery without knowing where it could come from.

The general agreement between Hansen's tables of 1857 with the work of Delaunay made it obvious that the three-body problem of Moon-Earth-Sun had finally been solved to the required precision. The most recent improvements of lunar theory by Hill and Adams fully supported the work of Hansen and Delaunay on the main problem, i.e., on the motions of the pure three-body system. The work of Newcomb on this important question can be followed in Archibald's bibliography (1924). Moreover, Hansen reported complete agreement with the observations from 1750–1850 to within errors of 1″ to 2″ at most. While this claim seemed correct, it came as a big surprise that the agreement deteriorated almost immediately to the point where some of the errors increased from 5″ by 1870, to 10″ by 1880, and 18″ by 1889.

The sources of these discrepancies are not easy to pin down. A major source of uncertainty arises from the acceleration of the Moon's motion, which was discovered by Halley and then explained by Laplace. But a careful recalculation by Adams produced a value only half of what was required by the observations. A judicious change in the parameters and in the initial conditions is not sufficient; the perturbations by the planets are not nearly large enough to bring the errors down. Finally an empirical inequality with a period of 273 years was recommended, without any attempt to explain its size or even its period.

In 1923 the Tables of Brown became the generally accepted basis for the computation of lunar positions, but their accuracy suffered a serious blow during the solar eclipse of January 24, 1925, mentioned in Sec. III.E. The path of totality crossed the upper half of Manhattan, with the southern boundary of the shadow at 96th Street. Consolidated Edison, New York City's electric utility company, posted 149 pairs of observers on the roofs along Riverside Drive on every block, in order to determine the exact line of the Moon's shadow (see TIES, 1925). Only one observer expressed doubt about the Sun's being covered by the Moon on his location. The line could be pinned down within 100 meters.

The New York Times carried a headline on the next day: Where was the Moon? It turns out that the Moon was four seconds late, not much by the standards of a casual observer, but much more than the accuracy that Brown had tried to achieve. An error of 4 time-seconds translates into an error of more than 2 arcseconds in longitude, roughly speaking. The precision of the shadowline, however, translates to an accuracy of better than 0.1 arcsecond in latitude. Actually the profile of the Moon varies by a few kilometers (Bailey's beads) with its exact orientation, and that may have been part of the problem with the longitude.

With the arrival of computing machinery in the 1930s the processing of data and the preparation of ephemerides could be accomplished without the incredible drudgery of earlier times. Celestial mechanics received an enormous impulse in the late 1950s from the sudden push for a large program of space exploration. Lunar theory in particular profited from President Kennedy's decision to have humans visit the Moon by the end of the 1960s. Indeed, there followed a prolific outpouring of interesting work, which will be reviewed somewhat summarily in this Section.

At the end of the 1990s it has to be admitted, however, that Tisserand's diagnosis is still valid. The main problem of lunar theory has been completely solved for all practical purposes, but there have been no major discoveries in getting a more direct analytical approach, and we are still wrestling with uncertainties in the comparison with the observations of almost 1 arcsecond. There is a long list of difficulties outside the strict confines of the three-body problem that have to be taken into account in order to improve the agreement with the observations. The various sections of this Section are intended to give the reader a brief glimpse of some of the efforts in this area.

Meanwhile the three-body problem in its full generality is still with us and occupies many volumes of papers in learned journals. Modern computers are the main resource for exploring all the possibilities of the general theory; there are many applications in the solar system and even some outside. But the Moon-Earth-Sun system is no longer among the active projects if one looks at the scientific literature.

B. The Moon's secular acceleration

In 1693 Halley compared the dates of some well-documented lunar eclipses: the longitude of the Moon could not be written as a linear function of time plus any number of sinusoidal corrections. A quadratic term in time was necessary to achieve a decent fit. Empirically, the lunar motion was found to suffer a small acceleration.

The longitude of the Moon can be represented schematically by the formula

$$\lambda=\lambda_0+nt+\sigma(t/c_{Jul})^2+\sum A\ \sin(\alpha t+\beta), \qquad (97)$$

where c_{Jul} designates one Julian century of 36525 mean solar days. The sum is extended over all the known periodic perturbations, from the Sun, the other planets, the shape of the Earth, and so on. The value of σ was determined during the 18th century by various astronomers to be about 10 arcseconds.

The glory of discovering the main cause belongs to Laplace (1787), who attributed it to the change in the eccentricity ε' of the Earth's orbit. The formal calculation is based on the second-to-last term of the first line in Eq. (53), i.e., the term proportional to ε'^2 in the perturbation W. The *Astronomical Almanac* for 1995 gives only the very rough formula $\varepsilon'=0.01671\,043-0.00000\,00012d$, where d, is the interval in days from 1995 January 0, 0 hours. The coefficient of the quadratic term in Eq. (97) is found to be 10.66 arcseconds.

Adams (1853, 1860) pursued this calculation to include higher corrections in the powers of m and found that all the higher powers in m decreased the value of the coefficient σ to 6.11″, too small by a factor of 2. It also became clear that the rotation of the Earth is bound to slow down because angular momentum gets transferred to the Moon due to the braking action of the tides. Part of the lunar acceleration is an ''optical illusion'' because it is really due to the slowing down of the Earth's rotation.

When the Moon acquires additional angular momentum, its potential energy with respect to the Earth increases, whereas its kinetic energy decreases. The Moon goes into a higher orbit where its speed with respect to the fixed stars decreases while its apparent speed with respect to the Earth increases. The slowing down of the Earth's rotation is well documented in prehistoric (geologic) times. The trouble in historic times is the irregular nature of this process, which finally forced the transition from the mean solar day to the mean solar year as the standard scale of time.

The ancient observations of eclipses, both lunar and solar, have been discussed with ever increasing sophistication for the purpose of determining the secular acceleration of the Moon as well as the rotation of the Earth; some of the more recent work is due to Spencer-Jones (1939), R. R. Newton (1970, 1979), Morrison (1978), and Stephenson (1978). Modern measurements are reported and critically assessed in monographs like that of Munk and MacDonald (1960), collections of special articles like those of Kopal (1961) and Brosche and Sündermann (1978), and Conference Proceedings like those edited by Marsden and Cameron (1966), Calame (1982), and Babcock and Wilkins (1988). The data are quite confusing to the layperson (see Van Flandern, 1970).

C. Planetary inequalities in the Moon's motion

The presence of the other planets in the solar system acts as an external disturbance. Their effect on the motion of the Moon is weak, so that they are generally investigated only after the main problem has been solved. Three ways to go about this task will be discussed briefly in this section: the variation of the constants of Lagrange and Poisson of Sec. VIII, the canonical formalism of Hamilton and Jacobi of Sec. IX, and the method of Hill and Brown that was explained in Sec. X.

It is unlikely that the Moon exerts any kind of perturbation on the planets of the solar system with the exception of the Earth; nor is the direct perturbation of the planets very effective. But Newton's proposition LXVI (see Sec. VI.E) reminds us that not only the planets, but the Sun itself has to respond to gravitational forces. Therefore we have to take into account the reaction of the Sun to the motion of the planet in its Kepler ellipse. As a particularly instructive example, Jupiter with 1/1000 solar mass is about 1000 solar radii away, so that the center of mass Sun-Jupiter lies just outside the Sun. This leads to a noticeable indirect effect on the Moon.

The first method is similar to the discussion in Sec. VIII.F concerning the effect of the Earth's shape on the Moon's orbit. The main job is to write down the energy of the gravitational interaction between the Moon and, say, Jupiter using the expansions of the Moon's longitude, latitude, and parallax in terms of the angles ℓ, g, h, ℓ' from Delaunay's work. The mass and the orbit of the planet are assumed to be known; its Kepler parameters are kept constant in first approximation. The planetary terms in addition to the solar perturbation Ω are inserted into Lagrange's equations (45)–(50).

By the end of the 19th century it became clear that the method of canonical transformations is by far the most efficient. Therefore the work of Delaunay on the solution of the three-body problem is the prerequisite. The planetary terms in the Hamiltonian become the object of some further canonical transformations, in von Zeipel's or any other form. Hill (1885) claims that "about ten days' work suffice for the elaboration" of Jupiter's action on the Moon. But the 16 large pages of arithmetic in Hill's *Collected Mathematical Works* do not look very encouraging!

D. Symplectic geometry in phase space

Whereas the Lagrange-Poisson method provides a modicum of physical insight at the price of some conceivable confusion, the canonical transformations on top of Delaunay's work are straightforward. The real challenge comes with extending the Hill-Brown results for the main problem of lunar theory to cover the perturbations due to the Earth's shape and due to the other planets.

The difficulty arises from Brown's decision to insert the numerical value of $m = n'/n$ from the very beginning, rather than to expand in powers of m as Delaunay and Hill did. The Moon's semi-major axis a determines her mean motion n in longitude according to Kepler's third law, and that parameter enters the solution of the main problem through m. The equations of motion, either in the Lagrange-Poisson form [Eqs. (45)–(50)] or in the canonical form [Eq. (63)], require the derivatives of the perturbation with respect to a. How can we find the derivative with respect to a parameter when only its numerical value is present in the solution?

Since the action L is directly related to the semi-major axis a by Eq. (62), the derivative with respect to L can be used. Brown (1903, 1908) invokes the symplectic nature of the trajectories in phase space to obtain the required derivatives of his solution with respect to the actions L, G, H. Unfortunately, this most imaginative application of the symplectic geometry of classical mechanics in phase space is not generally known. Meyer and Schmidt (1982) have given a much clearer and shorter version of Brown's original explanations.

Brown's method can be understood from looking at the special case in which we know the derivatives of the trajectory with respect to G, H, ℓ, g, h, but have used a numerical value for L. We write down explicitly the Poisson bracket,

$$[x, \dot{x}] = \frac{\partial x}{\partial L}\frac{\partial \dot{x}}{\partial l} - \frac{\partial \dot{x}}{\partial L}\frac{\partial x}{\partial l} + \frac{\partial(x, \dot{x})}{\partial(G, g)} + \frac{\partial(x, \dot{x})}{\partial(H, h)} = 1. \tag{98}$$

After dividing this equation with the known function $(\partial x/\partial \ell)^2$, the first two terms become simply the time derivative of $\partial x/\partial L$ divided by $\partial x/\partial \ell$. The value of $\partial x/\partial L$ is then obtained by integrating over time; the constant of integration is determined by the required symmetries and the other Poisson brackets.

Brown used the same idea to check the internal consistency of his solution. The Poisson brackets are more demanding than it appears at first, because their right-hand sides are constants. If the time derivatives are multiplied out, all the periodic terms have to cancel out. This check was also carried out on the computer by Schmidt (1980).

E. Lie transforms

As electronic computing became more efficient in the 1960s, and financial support for lunar theory became available, it was natural to emulate Delaunay's great feat with different means. A first try by Barton (1966) on the basis of a straightforward repetition, however, only got as far as calculating the perturbation function with the addition of ninth- and tenth-order terms. The canonical transformations themselves, even in von Zeipel's form, did not lend themselves easily to a systematic procedure that could be programmed. The expressions in the second line of Eq. (66) have to be inverted. In many practical cases this inversion depends on the possibility of expanding F as well as the Hamiltonian of the system in a power series with respect to a small parameter μ.

The method of Lie transforms avoids the inversion by aiming directly at the transformation that represents the old angles y and old actions Y in terms of the new angles x and new actions X. The transformation formulas are power series in μ where the zero order is the identity. The Lie derivative $L_F f$ of a function $f(y,Y)$ with respect to a generating function $F(y,Y)$ is the Poisson bracket $[F,f]$, and plays an essential role. The formalism is worked out in a fundamental paper by Deprit (1969).

The given Hamiltonian $\Omega(y,Y;\mu,t)$ is transformed into a particularly simple form such as $\Xi(x,X;\mu,t)$. The new Hamiltonian Ξ may be required to have no more terms in the angles x, at least up to a certain power of μ. At each step in the recursion, the function $F(y,Y)$ has to satisfy a first-order partial differential equation. This new scheme becomes a straightforward algorithm that can be programmed very efficiently. Once the generating function F is known, the same algorithm can be used to express the old action angles (y,Y) in terms of the new ones (x,X).

Many different detailed procedures were worked out in the late 1960s and early 1970s to realize and apply the Lie transforms in celestial mechanics, as well as to prove their equivalence. Hori (1963) seems to have been the first to try a new approach to lunar theory; see Stumpff (1974) for a more recent account. In particular, the relations between the formalisms of Hori (1966, 1967) and of Deprit (1969) are treated by Kamel (1969), Henrard (1970), Campbell and Jefferys (1970), Mersman (1970, 1971), Henrard and Roels (1973), Rapaport (1974), and Stumpff (1974). One application by Deprit and Rom (1970) that is closely related to the lunar problem concerns the main problem of satellite theory: a satellite circling the Earth is subject to the perturbation of the Earth's quadrupole moment. The most striking result of this method, however, is the complete recalculation of Delaunay's theory by Deprit, Henrard, and Rom (1971a), to be discussed in the next section.

F. New analytical solutions for the main problem of lunar theory

The exploration of the Moon by the U.S. National Aeronautics and Space Administration (NASA) during the 1960s and early 1970s provided the incentive for many celestial mechanicians to improve the available calculations of the Moon's motion. It is important to distinguish two types of solutions for the Moon's trajectory. So far we have discussed almost exclusively the main problem as defined at the end of Sec. VII.C, and its solution was always represented as a Fourier series in the four angles ℓ,g,h,ℓ'. They will be called "analytical" solutions to distinguish them from the "practical" solutions that are represented by a direct numerical integration of the equations of motion. The latter include right away the effect of all the other perturbations on the Moon besides the influence of the Sun.

A first effort went into completing some work that Airy (1889) had left unfinished and that had been criticized by Radau (1889). The idea is simple: the best available solution is substituted into the relevant equations of motion, and the necessary corrections are determined by varying the coefficients in the expansion of the solution. Eckert and Smith (1966) started from Brown's solution and achieved residuals in the 13th to 15th decimals by solving 10 000 equations of variation. Although the best computers of the time were run for several hundred hours, the report of the results still creates the impression of a tremendous effort in manual labor.

The next great enterprise was already mentioned at the end of Sec. IX because it tries to obtain a solution that is an improvement on Delaunay's classic work. It was called Analytical Lunar Ephemeris (ALE) by its creators at the Boeing Scientific Research Laboratories, Deprit, Henrard, and Rom (1970, 1971c). Regrettably, only certain parts of it have been reported in the scientific journals, presumably because the total output of data is too large. The method of Lie transforms was used to get a completely algebraic solution in all the parameters with all the coefficients as rational numbers. The work of Delaunay can now be used with confidence since the corrections are easily accessible.

Among the fascinating results is a short note by Deprit and Rom (1971) dealing with the long-period term in the Moon's longitude. It arises from the combination $3\lambda-\ell-2F$, whose period can be found from the figures in Sec. V.C to be 183 years. It leads to a denominator that Poincaré called "numériquement très petit" in contrast to the term "analytiquement très petit" that led to the disaster at the end of Sec. IX. The longitude was found to contain the term

$$\frac{315}{128}m\alpha\varepsilon\gamma^2\varepsilon' \sin(3l'-l-2F+3D), \tag{99}$$

which Laplace thought "quasi impossible" to predict from the theory. Its amplitude turns out to be completely negligible.

A project that is related to ALE was conceived by Henrard (1978, 1979) under the name of Semi-Analytical Lunar Ephemeris (SALE). It starts with a completely analytical solution of Hill's problem, i.e., the lunar trajectory in three dimensions if the Sun's perturbation is reduced to its average quadrupole field [Eq. (75)] in the neighborhood of the Earth. The solar eccentricity ε' and the ratio of the semi-major axes $\alpha = a/a'$ are neglected in this first step. Their effect is then treated as a perturbation to the fully algebraic solution of Hill's problem. The numerical agreement with ALE is excellent.

A completely independent approach was first conceived by Chapront-Touzé (1974, 1980) at the Bureau des Longitudes in Paris. The solutions are assumed to be the trigonometric series in the relevant angles, each with its own rate of change that has to be determined. The basic program has to manipulate these large series and then match their coefficients. In contrast to the projects mentioned so far, this effort has been pursued systematically to yield a complete ephemeris for the solar system under the abbreviation ELP 2000 because the various constants were adjusted to that epoch (see the further discussion in Sec. XI.J). The planetary perturbations, the effect of the Earth's and the Moon's shape, and even relativistic corrections are all taken into account, but the long-period term [Eq. (99)] is making trouble!

Finally, there is the work of Eckert that was mentioned at the end of Sec. X; it was based on Brown's development of Hill's approach to the lunar problem. Since the Lunar Laser Ranging (LLR) allowed the lunar distance to be measured with a precision approaching a few centimeters out of 400 000 kilometers, or 10^{-10}, it seemed reasonable to Gutzwiller and Schmidt (1986) that all terms in the expansion be calculated down to that level and be correct to 12 significant decimals. Such a requirement could be met if the calculations were made with so-called "double precision," which guarantees at least 14 decimals. The result would be called ELE for Eckert's Lunar Ephemeris.

As the work proceeded beyond Eckert's original plans, however, the term (99) made its appearance with a magnification by a factor of 2000 because of its small denominator. It was necessary to use "extended precision" and lower the cutoff to a level of 10^{-17}, which would correspond to a distance of a few nanometers on the Moon. That is physically quite absurd and betrays a mild form of chaos even in the Moon's motion. The author (Gutzwiller, 1979) showed, in the case of Eckert's original work, that the terms below a certain threshold create a noise whose root-mean-square of the amplitude is almost 10 times the threshold.

G. Extent and accuracy of the analytical solutions

Various modern methods for solving the main problem of lunar theory were described in the preceding section. Whereas SALE and ELE are at least partially analytic, ELP aims directly at finding the Fourier expansion of the lunar trajectory with purely numerical coefficients, not unlike Airy's method (see Eckert and Smith, 1966). All three calculations eventually yield the expansions for the polar coordinates of the Moon: longitude, latitude, and sine parallax. (The sine parallax is the ratio of the Earth's equatorial radius over the distance of the Moon, but this ratio becomes an angle if it is set equal to the sine of an angle.)

The cutoff for the listing in ELE was chosen at 0.000005″ for the longitude and latitude, which allows correct rounding to the fifth decimal. [The long-period inequality (99) of Laplace barely makes the grade with a coefficient 0.00000592″.] This threshold is to be compared with the largest terms, 22639.55″ $\sin l$ in longitude and 18461.40 $\sin F$ in latitude. They are generally agreed as the best fit to the observations and effectively define the eccentricity ε and the inclination γ (see the discussion in Sec. X.F.) The cutoff for the parallax is 0.00000 01″ and has to be compared (in this cosine expansion) with the constant term 3422.452″, which is again generally agreed to be the best observed value.

The cutoffs were chosen somewhat differently in SALE and ELP 2000, so that the total number of terms in the expansions for the polar coordinates is not exactly the same. For SALE, ELP 2000, and ELE there are 1177, 1024, and 1144 terms in longitude, 1026, 918, and 1037 terms in latitude, as well as 669, 921, and 915 terms in parallax. The agreement between ELP 2000 and ELE is practically perfect, with 1 coefficient in longitude differing by 2 in the last (fifth) decimal, 1 coefficient in latitude and 3 in sine parallax differing by 1 in the last decimal. The agreement with SALE in the last decimal has only a few more discrepancies. This almost complete coincidence between three large data sets that are based on entirely different computations shows that the main problem of lunar theory is solved correctly.

The reader might be interested in the distribution of the coefficients c in the Fourier expansions according to their size. Each bin is defined by its leading decimal when c is written in seconds of arc. For the lunar longitude and latitude we have, starting with $\log_{10} c \geqslant 4$, through $4 > \log_{10} c \geqslant 3$, all the way down to $-4 > \log_{10} c \geqslant -5$; for the sine parallax the counts are shifted downward by one bin. The number of coefficients in these bins are: for the longitude (1, 2, 10, 14, 32, 56, 192, 154, 268, 376), for the latitude (1, 1, 5, 7, 31, 49, 94, 151, 220, 357), and for the sine parallax (1, 1, 3, 4, 19, 28, 55, 96, 140, 238, 328) with one additional bin.

Although these counts cover 10 powers of 10, no simple model for the proliferation of terms in the Fourier series seems to work well. If the series are truncated by retaining only the terms with a coefficient above some threshold, the neglected terms generate a noise whose root-mean-square is larger than the threshold roughly by a factor 10 (Gutzwiller, 1979). The data were taken from the tables of Gutzwiller and Schmidt (1986), probably the last and most accurate record ever available in print.

H. The fruits of solving the main problem of lunar theory

Artificial satellites for the Earth and other planets, as well as a visit to the Moon by human beings, became a reality shortly after lunar theory had arrived at a sufficiently accurate and trustworthy solution of the three-body problem Moon-Earth-Sun. This sequence of events was no accident, of course; the scientific achievements in celestial mechanics and the technical progress in more mundane fields had grown simultaneously to the point where they could be joined in one big adventure. But nowadays the focus of general interest has completely shifted away from dealing with the complicated motion of the Moon; it is now concerned with the physical constitution of our companion in space.

Nevertheless, the astronauts of the Apollo program did some important work that is directly related to the Moon's motions. They left behind three "retroreflectors" that form a triangle with sides of 1250, 1100, and 970 km. The laser light that is sent to the Moon with the help mostly of the McDonald Observatory's 2.7-meter telescope (Silverberg, 1974) illuminates a spot of about 5 km in diameter. The reflectors send this light back exactly where it came from so that their signal is 10 to 100 times stronger than the reflected intensity from the lunar surface. By pulsing the light at rates of nanoseconds the distance of the reflectors can be measured with an accuracy of a few centimeters, a 10^{-10} fraction of the lunar distance. Two reflectors that were left on the Moon by unmanned Soviet vehicles returned the light only for a few days, possibly because of the dust from their vehicle.

The reports from this program, such as the one by Bender *et al.* (1973), make fascinating reading. Among the scientific objectives that were perceived in 1964 and that led to this Lunar Laser Ranging (LLR) experiment, is listed in first place "a much improved lunar orbit," in third place the "study of the lunar physical librations," and in fifth place "an accurate check on gravitational theory." This report already mentions a number of important advances in these three areas. French scientists were involved in the whole endeavor from the beginning (see Calame, 1973, and Orszag, 1973).

This section will give a very short summary of some results related to these scientific objectives. The most obvious improvement concerns the lunar parallax because its earlier measurements were always less direct than those of the longitude and latitude, and also more sensitive to the refraction of the Earth's atmosphere. Laser ranging avoids the triangulation that used to be the foundation for all distances in the solar system. The report by Bender *et al.* (1973) already quotes corrections to the lunar eccentricity, the mean longitude of the perigee, and the mean longitude of the lunar center of mass.

Everybody knows from first-hand experience that the Moon always turns the same side toward the Earth. Physically, the Moon rotates around its own axis at the same rate as it moves around the Earth. But the motion around the Earth is not uniform because of the eccentricity $\varepsilon \cong 1/18$ and the variation of Tycho Brahe. The Moon's orientation, however, is not coupled so strongly that it follows the direction of the Moon's center of mass. Moreover, we profit from the inclination of the lunar orbit and large parallax. The resulting changes in the Moon's appearance were well understood by the astronomers in the 17th century; they reveal almost an additional third of the remaining lunar surface.

In 1693 Giovanni Domenico Cassini published the following three laws:

(i) the Moon rotates at a constant angular velocity of one full rotation per sidereal month;

(ii) its axis is inclined by 2°30 with respect to the normal of the ecliptic (more exactly by 1°31 only);

(iii) the direction of the Moon's axis, the normal to the ecliptic, and the normal to the Moon's orbit lie in one plane.

In 1764 Lagrange won the prize of the French Academy in the competition to explain the libration of the Moon, but he succeeded only in explaining the equality of the mean motions in translation and rotation. He came back to the problem in 1780 to explain the coupling of the axes. Modern versions of this theory have been offered by Koziel (1962) and Moutsalas (1971), and further work presented in the volume edited by Kopal and Goudas (1967), as well as that edited by Chapront, Henrard, and Schmidt (1982).

With several reflecting telescopes on Earth sending laser pulses, not only the distance but also the relative orientations of the Earth and of the Moon can be determined. The centers of mass can be found and the figure of the Earth can be checked out. The Moon is almost a rigid body in contrast to the Earth. Not surprisingly, the rotational motion of the Earth is more complicated and of greater interest for us earthlings. A vast amount of work is reported in the scientific literature that ties in with LLR and with the dynamics of the Earth-Moon system generally [see the collections of articles edited by McCarthy and Pilkington (1979); Fedorov, Smith, and Bender (1980); and Calame (1982)].

All these articles deal with problems of great technical sophistication and are quite different in spirit from discussions of the Earth-Moon system like those collected by Marsden and Cameron (1966). Although the rotation of the Earth is again the central topic, the aim is an answer to questions concerning the long-term history and the constitution of our planet. At the same time, different views on the origin of the Moon are proposed, but the jury is still out on this issue after more than 30 years of deliberation. The Moon looks like an exception rather than a representative of the rule among the satellites in the solar system.

The last argument in favor of LLR was to check on gravitational theory. Indeed, the effects of general relativity on the motion of the Moon were discussed by de Sitter (1916) immediately after solving Einstein's equation for an isolated point-mass; Kottler (1922) gives an early review in the *Encyclopaedie der Mathematischen Wissenschaften*, where Pauli's review of general relativity was published. Among the many papers on general relativity in celestial mechanics, let me cite those of Finkelstein and Kreinovich (1976) and Mashhoon and Theiss (1991), who are particularly concerned with the Moon. Without trying to discuss their results, however, let me just make two comments of some historical interest.

A glance at a standard textbook on general relativity like that of Pauli (1921, 1958) shows that the Schwarzschild metric adds a short-range force with an inverse-fourth-power of the distance to the usual inverse-square force of gravitation. That is exactly what Clairaut tried to do in order to fudge the motion of the lunar perigee and get over Newton's frustrations (see Sec. VII.B). In a completely different context, L. H. Thomas had studied the effect of general relativity on the rotation of the Moon in the early 1920s, a problem he considered rather difficult. When he became interested in the spin of the electron, he remembered his work on the Moon; his famous factor 1/2 in the formula for the spin-orbit coupling of an electron then appeared without much effort, as he used to say (see Misner, Thorne, and Wheeler, 1972).

I. The modern ephemerides of the Moon

The Fourier (epicycle) expansions for the polar coordinates in the solar system were invented to make predictions before there was any physical understanding. That method continued to be useful even after Newton had shown that the problem is equivalent to integrating some ordinary differential equations. But the advent of electronic computing made it possible to integrate Newton's equations of motion without worrying about whether the solution has a good Fourier expansion or not. Moreover, the size of the problem could be enlarged to include many more than three bodies without much increasing the technical difficulties. The main limitation is the length of time over which the numerical integration is valid, whereas the Fourier expansion is not so limited.

The space agencies have come to rely on numerical integration because they allow us to handle the realistic circumstances, for example, to include the mass distribution in the Earth and the Moon in addition to the effect of the other planets and relativistic corrections. The result is a complete mathematical description of the lunar trajectory, exactly as it can be observed, not only as a mathematical model like the main problem of lunar theory. The various corrections to the main problem had been worked out to some extent ever since the 18th century, but it seems that there are so many complicated, although small, effects that the analytical solutions are unable to produce the required precision.

Jean Chapront and Michelle Chapront-Touzé (1982, 1983) have expanded their semianalytic solution of the main problem (see Sec. XI.F) into a full-fledged ephemeris, ELP 2000, for the Moon. The perturbations due to the planets, the shape of the Moon and the Earth, general relativity were all included, and the result was compared with the numerical integration of LE 200 from the Jet Propulsion Laboratory (JPL). The same authors (1991) also constructed a modern version of lunar tables that are still in the form of trigonometric series, but with the mean motions corrected with terms up to third and fourth power in time going beyond Eq. (97). Fewer terms are needed in the series, which is still able to represent the lunar motions from 4000 B.C. to A.D. 8000 although with some significant loss of accuracy far away from today.

For more than ten years, however, the ephemerides used in the various national Almanacs and for the work of the space agencies have been based on the numerical integrations from JPL and MIT/CFA. They have been developed in many steps of refinement ever since the 1970s, with the most recent version under the name DE 200/LE 200 (for development ephemeris and lunar ephemeris) in a series of papers by Standish (1982), Newhall (1983), Stumpff and Lieske (1984), and their collaborators. Some of this work is given in the list of references; it deals with difficult issues such as the practical definition of the inertial coordinate system, which cannot possibly be discussed in this review with any kind of depth. Nevertheless, some conclusions have to be mentioned because they show some of the essential differences between classical and modern astronomy.

Progress has not been as fast as one might think, however, since Calame (1982) still reported substantial disagreements between different numerical ephemerides. On the other hand, Kinoshita (1982) used numerical integration for the main problem of lunar theory to check on the Fourier series that were obtained by the theoreticians. It turned out that ELP, and to a slightly lesser degree SALE, essentially live up to their nominal accuracy, thus confirming their mutual agreement. Lieske (1968) analyzed 8639 observations of the minor planet Eros from 1893 to 1966 to get a better value for the mass of the Earth-Moon system as well as for the solar parallax. Soma (1985) went through thousands of lunar occultations from the years 1955 to 1980 to check up on ELP 2000. Standish (1990) examined many sets of optical observations to check on the accuracy of DE 200 for the outer planets.

There is no doubt in Standish's mind, however, that modern ephemerides for the Moon are best based on the LLR measurements (see Williams and Standish, 1989, 1990). Optical observations are tied to the star catalogues, which depend on the definition of the equator, the ecliptic, and their intersection in the equinox. The error in laser ranging depends on only the pulse width and the noise, but does not depend on the orientation of the reference frames. The Moon's longitude with respect to the Earth is good to 0.001″, and its mean motion to 0.04″/century where one Julian century covers $1.73\ 10^9{}''$. But in the long run there remains an uncertainty of $1''/cty^2$ in the secular acceleration. The agreement with the optical observations also suffers from a seemingly irreducible discrepancy of 1″.

J. Collisions in gravitational problems

Starting with Lagrange and through the first three quarters of the 19th century, it looked as if all the work in celestial mechanics could be reduced to one general method. Moreover, the source of all the difficulties appeared to be hidden in the three-body problem, of which the system Moon-Earth-Sun was the best-known example. There was hope that the general methods would provide some very fundamental insights and yield a satisfactory account of the general many-body problem.

This grand illusion was almost realized in the work of Sundman (1913). He first showed that triple collisions are not possible in the three-body problem if the total angular momentum does not vanish. Then he used the regularization of the double collision in the Kepler problem to show that such events do not destroy the smooth behavior of the three-body trajectory and its analytic dependence on time as well as on the other parameters. Finally he was able to construct a general solution that is analytic, i.e., it has a power-series expansion as a function of a timelike parameter (see Siegel and Moser, 1971). A tacit agreement seems to prevail among celestial mechanicians, however, that this result is useless because the relevant series converge very poorly (Diacu, 1996; Barrow-Green, 1996).

When two mass points have a near collision they make U turns around each other. The smooth limit of such an event, in which the two masses head straight for each other, is equivalent to the two bodies' bouncing off each other. Three-body collisions are quite different because there is no way to consider them as a smooth limiting case. When three bodies nearly collide at the same time, their kinetic energy and their (negative) potential energy are very large, although their sum is small. The scaling properties allow the problem to be reduced to the case where the total energy vanishes. The problem gets simplified, but the possibilities are still enormous (McGehee, 1974, 1975).

Three-body collisions can no longer be excluded in a gravitational problem with four or more bodies. Whenever one tries to classify the motions in a many-body system, it is crucial to understand what happens in a collision. If initial conditions or other parameters are changed continuously, the system may run into a collision. The two situations on either side of the collision have very little in common if more than two masses collide. Even the two-body collisions in the three-body problem are sufficiently complicated to prevent a complete classification of all three-body trajectories.

K. The three-body problem

Surveys of the general three-body problem are very bulky, e.g., Hagihara's *Celestial Mechanics* (1970–1976) in five volumes, where the last four volumes have two parts, each bound separately, and volume V with over 1500 pages is entitled "Topology of the Three-Body Problem." More recent monographs like *The Three-Body Problem* of Marchal (1990) are heavy on numerical calculations. The motion of the Moon has a significant overlap with the much smaller class of the restricted three-body problem, where two large masses move around each other on a fixed circular orbit, whereas the third mass is assumed to be so small that it does not interfere with the motion of the two large masses (Contopoulos 1966; Szebehely, 1967). Asteroids, space travel between Earth and Moon, and satellites of binary stars belong in this class. The problem is usually treated in two dimensions, so that Hill's motion of the lunar perigee becomes a special limiting case.

The main problem of lunar theory occupies a special niche in the space of all three-body problems, and there are many other niches of this kind. Since the Moon-Earth-Sun system is important to us and we have many precise observations, we have learnt a lot about this special niche and we have started to explore the others. But it is hard to carry our experience from the Moon-Earth-Sun corner, say, into one of the asteroid corners. When the mass ratios, the main frequencies, and the initial conditions are modified so as to get from one niche to the other, we shall find out that our solutions have only limited validity.

The most dramatic event in a three-body system is the ejection of one mass, e.g., the Earth could get a little closer to the Sun and thereby provide the Moon with the required energy for escape, from the Earth if not from the solar system. Such ejection trajectories are hard to come by, but some people believe that they are everywhere dense in phase space, like a fractal of very low dimension. Indeed, the proliferation of terms in the Fourier expansion of the lunar motion is not only a big nuisance, but could cover up what is known as Arnold diffusion.

If the motion of the Moon is restricted to the ecliptic, her trajectory can be viewed as filling a two-dimensional torus in a three-dimensional phase-space, as long as the Earth-Moon planet goes around the Sun on a circle (see Sec. X.F.) If this circle is replaced by a more realistic Kepler ellipse, the torus will breathe with the yearly rhythm. Every invariant torus divides the phase space into separate pieces, and any chaotic trajectory that starts inside a torus cannot escape. If the Moon is allowed to move at a right angle to the ecliptic, however, her invariant torus is three-dimensional, while the phase space gets five dimensions. Now, a chaotic trajectory is no longer caught, although the escape in this kind of Arnold diffusion always takes a very long time.

Poincaré was the first to make us think along these lines. We have been able to amass a large store of individual investigations in the last hundred years, most of them coming from computers in the last few decades. But we are still missing a general approach that allows us to understand all the different kinds of behavior from one common point of view.

The three-body problem teaches us a sobering lesson about our ability to comprehend the outside world in terms of a few basic mathematical relations. Many physicists, maybe early in their careers, had hopes of coordinating their field of interest, if not all of physics, into some overall rational scheme. The more complicated situations could then be reduced to some simpler models in which all phenomena would find their explanation. This ideal goal of the scientific enterprise has been promoted by many distinguished scientists [see Weinberg's (1992) *Dream of a Final Theory*, with a chapter "Two Cheers for Reductionism"].

Newton saw the motion of the planets in the solar system as instances of the two-body problem. The observed changes in their aphelia and eccentricities as well as in the nodes and inclinations of their orbital planes could only be detected after collecting data with instruments over many years. The Moon is different: any alert individual could watch her motion through the sky and become aware of her idiosyncracies. Both her varying speed and the spread of moonrises and moonsets on the horizon proceed at their own rhythm, which is most clearly displayed in the schedule of lunar and solar eclipses. These rather obvious features are the most important manifestations of the three-body problem and became the first objects of Newton's attention.

Many physicists may be tempted to see in Newton's equations of motion and his universal gravitation a sufficient explanation for the three-body problem, with the details to be worked out by the technicians. But even a close look at the differential equations (29) and (30) does not prepare us for the idiosyncracies of the lunar motion, nor does it help us to understand the orbits of asteroids in the combined gravitational field of the Sun and Jupiter. This review was meant to demonstrate the long process of trial and error, including many intermediate stops, that finally led to the modern lunar theory. The reader is expected to make a choice among the many different pictures and explanations.

Since the equations of motion (29) and (30) can be integrated on a computer nowadays, an option that was not available a generation ago, they are sufficient for some special purposes. The expansions (93) and (94) give the whole story, but only if they are interpreted in the light of some specific question or in order to carry out some particular task. If we want to appreciate the tremendous amount of information in these series, we are almost forced to fall back on the earlier treatments of the problem. They used a limited and perhaps primitive, but also more explicit and direct approach that made them more easily understood.

LIST OF SYMBOLS

A	IV.F	aphelion or apogee
D	V.C	$\lambda-\lambda'$=elongation (Moon from Sun as seen from Earth)
D	X.C	operator $d/id\tau$ in Hill' theory
E	IV.F	Earth (mass or location)
F	V.B	$l+g$=argument of latitude
F	VII.D	angular momentum parameter in Clairaut's theory
F	IX.C	generating function for canonical transformation
$\vec{F}$	V.B	Runge-Lenz vector for Moon's motion around Earth
G	IX.B	action variable conjugate to g
G_0	II.J	gravitational constant
H	IX.B	action variable conjugate to h
K	II.C	pole of the ecliptic
K	IX.C	action variable conjugate to k
L	IX.B	action variable conjugate to l
$\vec{L}$	V.B	angular momentum of Moon with respect to Earth
M	V.C	Moon (mass or location)
$M(\zeta)$	X.D	function in the Moon's variational equation
N	II.B	north pole
$N(\zeta)$	X.D	function in the Moon's variational equation
O	IV.E	center of the planet's orbit
P	II.B	north pole
P	IV.F	perihelion or perigee
Q	II.C	reference point on equator or eclipse (equinox)
R	VI.E	distance from Sun to center of mass Earth-Moon
S	IV.E	Sun (mass or location)
S'	IV.E	equant point for planetary motion
T	V.C	synodic month (Sun to Sun)
T_0	V.C	tropical (sidereal) year (equinox to equinox)
T_1	V.C	tropical (sidereal) month (equinox to equinox)
T_2	V.C	anomalistic month (radial motion)
T_3	V.C	draconitic month (motion vertical to ecliptic)
V	X.B	quadrupole potential of the Sun near the Earth
W	VIII.C	perturbing solar potential
$\vec{X}$	VI.E	vector from S to Γ
Z	II.B	local zenith
a	V.C	semi-major axis of Moon with respect to Earth
a'	V.C	semi-major axis of Earth-Moon with respect to Sun
c_0	X.D	Hill's lowest-order motion of the perigee
c_{Jul}	XI.B	Julian century=36525 days
f	V.A	Moon's true anomaly (perigee to Moon)
f'	IV.G	planet's true anomaly (perihelion to planet)
g	V.A	angle from Moon's ascending node to perigee
g_0	X.E	Hill's lowest-order motion of the node
$g'+h'$	VIII.G	angle from Q to perihelion
h	V.A	angle from reference Q to ascending node
k	VII.D	Clairaut's semi-major axis parameter
k	IX.C	$=f'+g'+h'$ (mean longitude of the Sun)
l	V.B	Moon's mean anomaly with respect to Earth
l'	IV.G	planet's mean anomaly with respect to Sun
l_0	VIII.B	Moon's mean anomaly at $t=0$ (epoch)
m	VII.C	n'/n=ratio of solar to lunar mean motion
m'	X.C	$n'/(n-n')$=ratio of solar to synodic mean motion
n	V.C	lunar mean motion$=2\pi/T_1$
n'	V.C	solar (Earth's) mean motion$=2\pi/T_0$
$n-n'$	X.C	synodic mean motion$=2\pi/T$
r	VI.E	distance from Earth to Moon
r'	VI.E	distance from Sun to Earth

s	VII.C	lunar parallax (inverse distance) in Clairaut's theory
t_0	IV.G	time of perigee (or perihelion) passage
u	IV.D	$x+iy=$complex coordinate in ecliptic
v	IV.H	eccentric anomaly (angle from the center of the orbit)
$\vec{x}$	VI.E	(x,y,z) Moon's coordinates in the ecliptic with respect to E
$\vec{x}\,'$	VI.E	(x',y',z') Earth's coordinates in the ecliptic with respect to Sun
Γ	VI.E	center of mass for the Earth-Moon system
Δ	X.D	Hill's infinite determinant for the motion of the perigee
Φ	VIII.E	polar coordinate for Γ around the Sun
Ψ	VI.C	total angle from one perigee to the next
Ω	VII.C	perturbation in Clairaut's theory
Ω	IX.B	total Hamiltonian for the Moon-Earth-Sun system
α	II.C	right ascension
α	X.G	$a/a'=$distance of Moon/distance of Sun
β	II.C	latitude
β	VII.E	amplitude of Tycho Brahe's variation
γ	V.A	inclination of the Moon's orbital plane
δ	II.B	declination
δ	V.D	amplitude of Ptolemy's evection
ε	V.B	eccentricity (of the Moon's orbit around the Earth)
ε'	IV.C	eccentricity (of the Earth-Moon's orbit)
ε_0	II.C	obliquity of the ecliptic
$\vec{\xi}$	VI.E	vector from Sun to Moon
ζ	II.B	altitude
ζ	X.C	exponential $\exp(-i\tau)$ in Hill's theory
λ	II.C	longitude of the Moon (later$=\ell+g+h$)
λ'	V.C	longitude of the Sun (later$=\ell'+g'+h'$)
μ	VII.C	rate of angular increase for perigee
μ	IX.B	reduced mass of Earth-Moon $EM/(E+M)$
ν	VII.K	rate of angular increase for node
ρ	VI.E	distance from Sun to Moon
τ	II.C	sidereal time
τ	X.C	time variable in normalized synodic motion
ϕ	II.B	geographic latitude
ϕ	VII.C	Moon's polar coordinate in $(r,\ \theta,\ \phi)$
ψ	II.B	azimuth
χ	II.B	hour angle

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