Stability of Non-Neutral Plasma Cylinder Consisting of Magnetized Cold Electrons and of Small Density Fraction of Ions Born at Rest: Non-Local Analysis

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Model of Plasma Considered

Geometry of problem

Magnetized Electrons \( \rho_{Le} \ll a \)

- \( n_e = \text{const}, \quad r < a, \quad \omega_e = \text{const} \)

- \( \varepsilon_\perp = e\Phi(r_0), \quad M = \frac{m_i\omega_{ci}r_0^2}{2}, \quad v_z = 0 \)

Ions Born at rest: \( \vec{v}_0 = 0 \)

Ions Can Be Unmagnetized

Can Be Magnetized

Radial Electric Field \( E_r \) is related to the charge imbalance

\[-\frac{e}{m_e} E_r = \frac{\omega_{pe}^2 (1 - f)}{2} r, \quad f = \frac{N_i}{n_e} \ll 1 \]

- it can be strong for ions \( \omega_{ci}^2 \ll 4eE_r / m_i r \)

- it can be even of the order of critical field for electrons \( \omega_{ce}^2 \geq 4eE_r / m_e r \)

- can be weak

There is no vacuum gap between plasma and metal!

Electric Potential :

\( \Phi(r) = \Phi(a) r^2 / a^2 \)
Where Can Such Ions Be Born?

- Plasma Lenses, Plasma Ion Sources, based on Penning cell Traps;
- Electron and Ion Beams (Secondary Plasmas in Beam Channel)
- «Penning fusion» Project;
- Open Confinement System (PSP-2, Novosibirsk);
- CNT-?
- C-shaped Trap ("Partial" Torus)?
The amplitude of ion radial oscillations can be of the order of plasma radius

The Non-Local Problem about the Stability of Non-Neutral Plasma, should be considered on the ground of kinetic treatment of ions.

\[ \omega_c^2 = \frac{4e(E_r)_{\text{crit}}}{m_i r} \]

Vlasov equation for ions + Poisson equation

Potential of plasma oscillations:

\[ \tilde{\Phi}_m (r) = \sum_{l=1}^{\infty} C_m J_m \left( \kappa_{m,l} \frac{r}{a} \right) / N_m^l \]

\[ N_m^l = (1/\sqrt{2}) \left| J_{|m|+1} (\kappa_{m,l}) \right| \]
Method:
Solution of Vlasov equation for ions in variables, corresponding to cylindrical coordinates of Larmor centre and coordinates of particle on Larmor circle \( R, \theta, \rho, \vartheta \) is used.

\[
r \exp(i\varphi) = R \exp(i\theta) + \rho \exp(i\vartheta) = \exp(i\omega_+ t) \left[ R \exp(i\theta_0) + \rho \exp(i\vartheta_0 - \Omega_i t) \right]
\]

\[
\theta = \theta_0 + \omega_+ t, \quad \vartheta = \vartheta_0 + \omega_- t, \quad \omega_\pm = (-\omega_c \pm \Omega_i)/2, \quad \Omega_i = \left( \omega_c^2 - 4eE_r / m_i r \right)^{1/2}
\]

\[
\tilde{f} = -\frac{e}{m_i} \sum_{l=1}^{\infty} \sum_{p=-\infty}^{\infty} C_m^l (-1)^p \frac{1}{N_m^l} J_{m+p}(\kappa_m, \frac{R}{a}) J_p(\kappa_m, \frac{\rho}{a}) \cdot F - ?
\]

\[
\begin{bmatrix}
    m + p & \frac{\partial F}{\Omega_i R \partial R} & p & \frac{\partial F}{\Omega_i \rho \partial \rho} & k_z \frac{\partial F}{\partial v_z} \\
    \omega_+ & \omega_i & \omega_- & -k_z v_z \\
\end{bmatrix}
\exp \left[ i((m + p)\theta - p\vartheta + k_z z - \omega t) \right]
\frac{\omega - (m + p)\omega_+ + p\omega_- - k_z v_z}{\omega - (m + p)\omega_+ + p\omega_- - k_z v_z}
\]

Method developed by
Compare the form of solution for $\tilde{f}$ in different variables!

$$\tilde{f} = -\frac{e}{m_i} \sum_{l=1}^{\infty} \sum_{p=-\infty}^{\infty} C_m^l (-1)^p \frac{1}{N_m^l} J_{m+p} (\kappa_{m,l} \frac{R}{a}) J_{m} (\kappa_{m,l} \frac{\rho}{a}) \cdot \exp \left[ i((m+p)\theta - p\dot{\theta} + k_z z - \omega t) \right]$$

$$\times \Omega_i \left( \frac{m+p}{R \partial R} \frac{\partial F}{\Omega_i \rho \partial \rho} + k_z \frac{\partial F}{\partial v_z} \right) \cdot \frac{\exp \left[ \omega - (m+p)\omega_+ + p\omega_- k_z v_z \right]}{\omega - (m+p)\omega_+ + p\omega_- k_z v_z}$$


$$\tilde{f} = e \frac{\partial F}{\partial \varepsilon_\perp} \tilde{\Phi}_m (r) + e \frac{\exp (i \delta_\tau)}{\sin \delta_\tau} \left[ (\omega - k_z v_z) \frac{\partial F}{\partial \varepsilon_\perp} + m \frac{\partial F}{\partial M} + \frac{k_z}{m_i} \frac{\partial F}{\partial v_z} \right] \times$$

$$\times \int_{r_{\min}}^{r_{\max}} \frac{dr' r' \tilde{\Phi}_m (r') \exp (-i \delta)}{r_{\min} \left[ \frac{2}{m_i} (\varepsilon_\perp - e \Phi (r')) r'^2 - \left( \frac{M}{m_i} - \frac{\omega_{ci} x^2}{2} \right) \right]^{1/2}} \cdot$$

$$\delta_\tau = (\omega - k_z v_z) \tau - m \phi_\tau,$$

$$\delta = \int \frac{dxx \left[ \omega - k_z v_z - m \left( \frac{M}{(m_i x^2)} - \frac{\omega_{ci}}{2} \right) \right]}{r \left[ \frac{2}{m_i} (\varepsilon_\perp - e \Phi (x)) x^2 - \left( \frac{M}{m_i} - \frac{\omega_{ci} x^2}{2} \right)^2 \right]^{1/2}} \cdot$$

**Ion equilibrium distribution function** has been unequivocally determined by condition of birth of ions at rest

\[
F(\varepsilon_\perp, M, v_z) = \frac{N}{T} \frac{m_i}{\omega_{ci}} Y(e\Phi(a) - \varepsilon_\perp) \cdot \delta(\varepsilon_\perp - \omega_c M) \cdot \delta(v_z)
\]

is caused by the absence of ions with energy \( \varepsilon_\perp > e\Phi(a) \) and makes the distribution function \( F \) similar to Fermi – Dirac function

\[
T = 2\pi / \Omega_i, \quad \Omega_i = \left( \omega_{ci}^2 - 4eE_r / m_i r \right)^{1/2} = \left( \omega_{ci}^2 + 8e\Phi(a) / m_ia^2 \right)^{1/2}
\]

- “modified” ion cyclotron frequency

\[
r < r_{\text{min}}(a) = a\omega_{ci} / \Omega_i
\]

- “rigid rotator” type!

It does not belong to “rigid rotator” type!

is caused by the birth of ions at rest and makes the distribution function similar to “rigid rotator” type with frequency

\[
\omega_{\text{rot}} = \omega_e = - cE_r / Br = (1 / 2)(\omega_{pe}^2 / \omega_{ce})!
\]

\[\Phi(r) = \Phi(a) r^2 / a^2\]

\[r > r_{\text{min}}(a) = a\omega_{ci} / \Omega_i\]

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**Ion equilibrium distribution function**

Its form in variables $\varepsilon_\perp, M, \nu \perp$

$$F(\varepsilon_\perp, M, \nu_\perp) = \frac{N}{T} \frac{m_i}{\omega_{ci}} Y \left( e\Phi(a) - \varepsilon_\perp \right) \cdot \delta \left( \varepsilon_\perp - \omega e M \right) \cdot \delta \left( \nu_\perp \right)$$

Its form in chosen variables $R, \theta, \rho, \vartheta$:

$$\varepsilon_\perp = \frac{m_i}{2} \Omega_i \left( R^2 \omega_+ - \rho^2 \omega_- \right), \quad M = \frac{m_i}{2} \Omega_i \left( R^2 - \rho^2 \right)$$

$$F(R, \rho, \nu_z) = \frac{N}{\pi} Y \left( 1 - R^2 \frac{\Omega_i}{(a^2 |\omega_i|)} - \rho^2 \frac{\Omega_i}{(a^2 |\omega_i|)} \right) \cdot \delta \left( \rho^2 \omega_-^2 - R^2 \omega_+^2 \right) \cdot \delta(\nu_z).$$

$$\omega_\pm = (-\omega_{ci} \pm \Omega_i) / 2, \quad \Omega_i = \left( \omega_{ci}^2 - 4eE_r / m_i r \right)^{1/2}$$

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\[ F(R, \rho, v_z) = \frac{N}{\pi} \left( 1 - R^2 \frac{\Omega_i}{(a^2|\omega_i|)} - \rho^2 \frac{\Omega_i}{(a^2|\omega_i|)} \right) \cdot \delta \left( \rho^2 \omega^2 - R^2 \omega^2 \right) \cdot \delta(v_z). \]

\[ \tilde{f} = -\frac{e}{m_i} \sum_{l=1}^{\infty} \sum_{p=\infty}^{\infty} C_m (-1)^p \frac{1}{N_m} J_{m+p} \left( \frac{R}{a} \right) J_p \left( \frac{\rho}{a} \right) \cdot \left\{ \frac{m + p}{\Omega_i} \frac{\partial F}{R \partial R} + \frac{p}{\Omega_i} \frac{\partial F}{\rho \partial \rho} + k_z \frac{\partial F}{\partial v_z} \right\} \cdot \exp \left[ i((m + p)\theta - p\vartheta + k_z z - \omega t) \right] \cdot \omega - (m + p)\omega + p\omega - k_z v_z \]

\[ \tilde{\rho}_i = \int \tilde{f} d^3v \]

**Poisson equation**

\[ \int (\tilde{\rho}_i, \Delta \tilde{\Phi}) J_m \left( \kappa_{m,k} r/a \right) rdr \]

**Integro-differential equation**

\[ \left( L_{kl} - A_{kl} \right) C_m^l = 0 \]

- Integrating, we obtain a set of linear equations

\[ d^3r \cdot d^3v = \Omega_i^2 R \rho \cdot dR d\theta d\vartheta d\rho d\varphi dz dv_z \]

We know.

**Expansion in Bessel Functions**

\[ \tilde{\Phi}_m(r) = \sum_{l=1}^{\infty} C_m^l J_m \left( \kappa_{m,l} r/a \right) / N_m \]
In this report we consider «Cold» electrons:

\[ (L_{kl} - A_{kl})C_m^l = 0 \]

\[ L_{kl} = (\kappa^2_{m,k} \varepsilon_1^e + k_z^2 a^2 \varepsilon_3^e) \delta_{kl} = \text{Electron contribution + Laplacian of } \tilde{\Phi} \]

\[ \varepsilon_1^e = 1 - \frac{\omega_{pe}^2}{(\omega - m\omega_e)^2 - \Omega_e^2}, \quad \varepsilon_3^e = 1 - \frac{\omega_{pe}^2}{(\omega - m\omega_e)^2} \]

\[ \omega_e = \frac{1}{2} \left( |\omega_{ce}| - \Omega_e \right), \quad \Omega_e = |\omega_{ce}| \left( 1 - \frac{2\omega_{pe}^2}{\omega_{ce}^2} (1 - f) \right)^{1/2} \]

\[ f = \frac{N}{n_e} = \text{const} \quad \text{- fractional neutralization of electron cloud} \]


\[ A_{lk} = \text{Contribution of ions} \]

Parameter of our problem

\[ \frac{m\omega_e}{k_z^2 n_{Te}} >> 1 \quad \frac{2\omega_{pe}^2}{\omega_{ce}^2} >> \frac{4}{m(1 - f)} k_z \rho_e \]
Contribution of Ions in Poisson Equation

\[ A_{kl} = \frac{\omega_{pi}^2}{N_m^l N_m^k} \sum_{p=-\infty}^{\infty} \left\{ \frac{1}{\omega_+^2} \left( \frac{p}{\omega'} \right) \right\} J_{m+p}(z_i^-) J_{m+p}(z_k^-) J_p(z_i^+) J_p(z_k^+) + \]

\[ + \left[ \frac{m}{\omega_-^2} + p \left( \frac{1}{\omega_-^2} - \frac{1}{\omega_+^2} \right) \right] \cdot \frac{1}{\omega'} \left( \frac{\omega'}{\Omega_i} - p \right) \int_0^1 dx \frac{d}{dx} \left[ J_{m+p}(z_i^- x) J_{m+p}(z_k^- x) \right] J_p(z_i^+ x) J_p(z_k^+ x) + \]

\[ + \frac{1}{\Omega_i^2} \left( \frac{\omega'}{\Omega_i} - p \right) \int_0^1 x dx J_{m+p}(z_i^- x) J_{m+p}(z_k^- x) J_p(z_i^+ x) J_p(z_k^+ x) \right\}, \]

This expression is valid at arbitrary strengths of fields!

\[ \omega' = \omega - m \omega_+, \]

\[ z_i^{\pm} = \kappa_{m,l} \left| \omega \right| / \Omega_i, \]

\[ \omega_{\pm} = (-\omega_{ci} \pm \Omega_i) / 2, \]

\[ \Omega_i = \left( \omega_{ci}^2 - 4 e E / m_i r \right)^{1/2} \]

Without ions: \[ \det(L_{kl} - A_{kl}) = 0 - \text{Trivelpiece - Gould Modes in NNP} \]

\[ \omega = m \omega_e \pm \left\{ \frac{\Omega \frac{2}{e} + \omega_0^2}{2} \pm \left[ \frac{\Omega \frac{2}{e} + \omega_0^2}{2} \right]^2 - \cos^2 \theta \omega_0^2 \right\}^{1/2} \]


\[ \cos^2 \theta = k_{\perp}^2 \left( k_{\parallel}^2 + k_{\perp}^2 \right), \quad k_{\perp}^2 = \frac{\kappa^2}{m}, l / a^2 \]

In Neutral Plasma: dependences on \( k_z \)

Neutral Resting Plasma

\[ \omega' \rightarrow \omega \]

Neutral Drifting Plasma

\[ \Omega \rightarrow \omega_{ci} \]

In Non-Neutral Plasma:

In non-neutral plasma, the parameter of our problem is given by:

\[
\frac{2\omega_{pe}^2}{\omega_{ce}^2}(1 - f) = \frac{4|eE_r|}{m_e r \omega_{ce}^2} = \frac{4\omega_e}{\omega_{ce}} \leq 1
\]

Critical Radial Electric Fields (\(a = 0.5 \text{ cm}\))

The Brillouin limit for electrons in strong electric fields and critical radial electric fields for ions are shown in the graph. The graph illustrates the behavior of the parameter \(\Phi(a, V)\) as a function of the magnetic field \(B\) and the density ratio \(n_2/n_1\).

Critical radial electric fields for weak electric fields and «Cold» Electrons are also depicted.


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In Non-Neutral Plasma: **Trivelpiece - Gould Modes**

Dependences on the parameter

\[ \omega = m \omega_e \pm \left\{ \frac{\Omega_e^2 + \omega_{pe}^2}{2} \pm \left[ \frac{\Omega_e^2 + \omega_{pe}^2}{2} \right]^2 - \cos^2 \theta \omega_{pe}^2 \Omega_e^2 \right\}^{1/2} \]

«Cold» Electrons

\[ \frac{2 \omega_{pe}^2}{\omega_{ce}^2} \]

In details:

\[ \omega/\omega_{ce} (GT+) \]

\[ \omega/\omega_{ce} (GT-) \]

\[ 2 \omega_{pe}^2 / \omega_{ce}^2 \]
Results of our computations \[ \det(L_{kl} - A_{kl}) = 0 \] «Cold» Electrons

A small amount of ions is present in plasma

\[ m = 1; \quad k_z a = 0,1; \quad f = N / n_e = 0,01; \quad \text{nitrogen} \]

Graphical representation showing various modes and their frequencies.
\[ \omega' / \Omega = \omega - m \omega_+ , \]
\[ \omega_\pm = \left( -\omega_e \pm \Omega_i \right) / 2, \quad \Omega_i = \left( \omega_e^2 - 4 e E_i / m_r \right)^{1/2} \]
\[ \varepsilon_1^c = 1 - \frac{\omega_{pe}^2}{(\omega - m \omega_e)^2 - \Omega_e^2} \]

Results of computations \[ \det(L_{kl} - A_{kl}) = 0 \]
Results of computations $\det(L_{kl} - A_{kl}) = 0$

«Cold» Electrons

$\omega' = \omega - m\omega_+$,

$\omega_+ = (-\omega_i \pm \Omega_i)^2 / 2$, $\Omega_i = (\omega_i^2 - 4eE_i / m_r)^{1/2}$

$\varepsilon_1^e = 1 - \frac{\omega pe}{(\omega - m\omega_+)^2 - \Omega^2}$

$\text{Re}\omega/\Omega = -1$

$\text{Im}\omega/\Omega$

$2\omega^2 \omega_c^2$

$\omega'/\Omega$

$\text{Re}\omega'/\Omega$

$\text{Im}\omega'/\Omega$

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Results of computations \( \det(L_{kl} - A_{kl}) = 0 \)

«Cold» Electrons

\[ m = 1, \ k_z = 0.1, \ N/n_e = 0.01 \]

\[ 2 \omega_{pe}^2/\omega_{ce}^2 \]

\[ \omega' / \Omega, \ \text{Re}(\omega'/\Omega) \]

\[ m = 1, \ k_z = 0.1, \ N/n_e = 0.01 \]
Results of computations \( \det(L_{kl} - A_{kl}) = 0 \)

\[ m=1, \; k_z=0.1, \; N/n_e=0.01 \]

«Cold» Electrons

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Results of computations \( \det(L_{kl} - A_{kl}) = 0 \)

Digits denote maximum normalized growth rates of TG modes

Here MICW are unstable only with maximum growth rates shown

\( \gamma_{\text{max}} \approx 0.074 \)

\( \Omega_i \)

«Cold» Electrons

\( m = 1; \ k_z a = 0.1; \ f = N / n_e = 0.01; \) nitrogen

\( \omega' / \Omega \)

\( \omega = \omega_m \omega_+; \ \frac{\omega_+}{2} = \frac{\omega_m + \Omega_i}{2}; \ \Omega_i = \left( \frac{\omega^2_m - 4eE_i}{m_i r} \right)^{1/2} \)

\( 2\omega_{pe}^2 / \omega_{ce}^2 \)
Peculiarities of Obtained Spectra

1. Oscillations on Frequency Dependencies of “Modified” Ion Cyclotron Waves. They are similar to oscillations on dispersion curves of electron waves in metals and are caused by the similarity of the equilibrium distribution function of ions with the degenerate Fermi-Dirac distribution function.


2. Frequency Jumps under Continuous Changing of Fields

Re(ω'/Ω_i)
Summary

✓ The non-local problem is solved about the stability of non-neutral plasma cylinder, consisting of “cold” magnetized electrons and small amount of ions. The consideration is based on kinetic treatment of ions and takes into account the equilibrium distribution function of ions, born at rest.

The considered problem is a special case of problem, stated by R.H. Levy, J.D. Daugherty and O. Buneman [Phys. Fl. 1969. V. 12. P. 2616], when plasma bounds directly with metal casing (vacuum gap between plasma and metal is absent, so only volumetric eigen frequencies exist).

✓ The dispersion equation is obtained analytically! It is valid within the whole allowable range of values of field strength, for magnetized and unmagnetized ions!

✓ The spectra of eigen oscillations are found numerically.
Summary

✔ Two types of waves exist in considered plasma model:
  ▪ electron Trivelpiece - Gould (TG) modes,
  ▪ “modified” ion cyclotron waves (MICW).

✔ The frequencies of Trivelpiece - Gould modes in non-neutral plasma fall into the region of ion frequencies.
The slow modes TG are unstable in the vicinities of non-negative harmonics of “modified” ion cyclotron frequency \( \Omega_i \).
Maximum growth rate is achieved at crossing with zero harmonic:
\[
\gamma \approx 0.074 \Omega_i \quad (N_i / n_e = 0.01)
\]

✔ The “modified” ion cyclotron waves form families, located in a small vicinity of harmonics \( \Omega_i \) above and below the harmonic.
Their growth rates are small \( (\text{Im } \omega' / \Omega_i)_{\text{max}} \leq 0.002 \).

✔ On dependencies of MICW frequencies the oscillations are seen. They are similar to oscillations on dispersion curves of electron waves in metals.
Thank You