The Empirical Mode Decomposition and Hilbert Spectrum.

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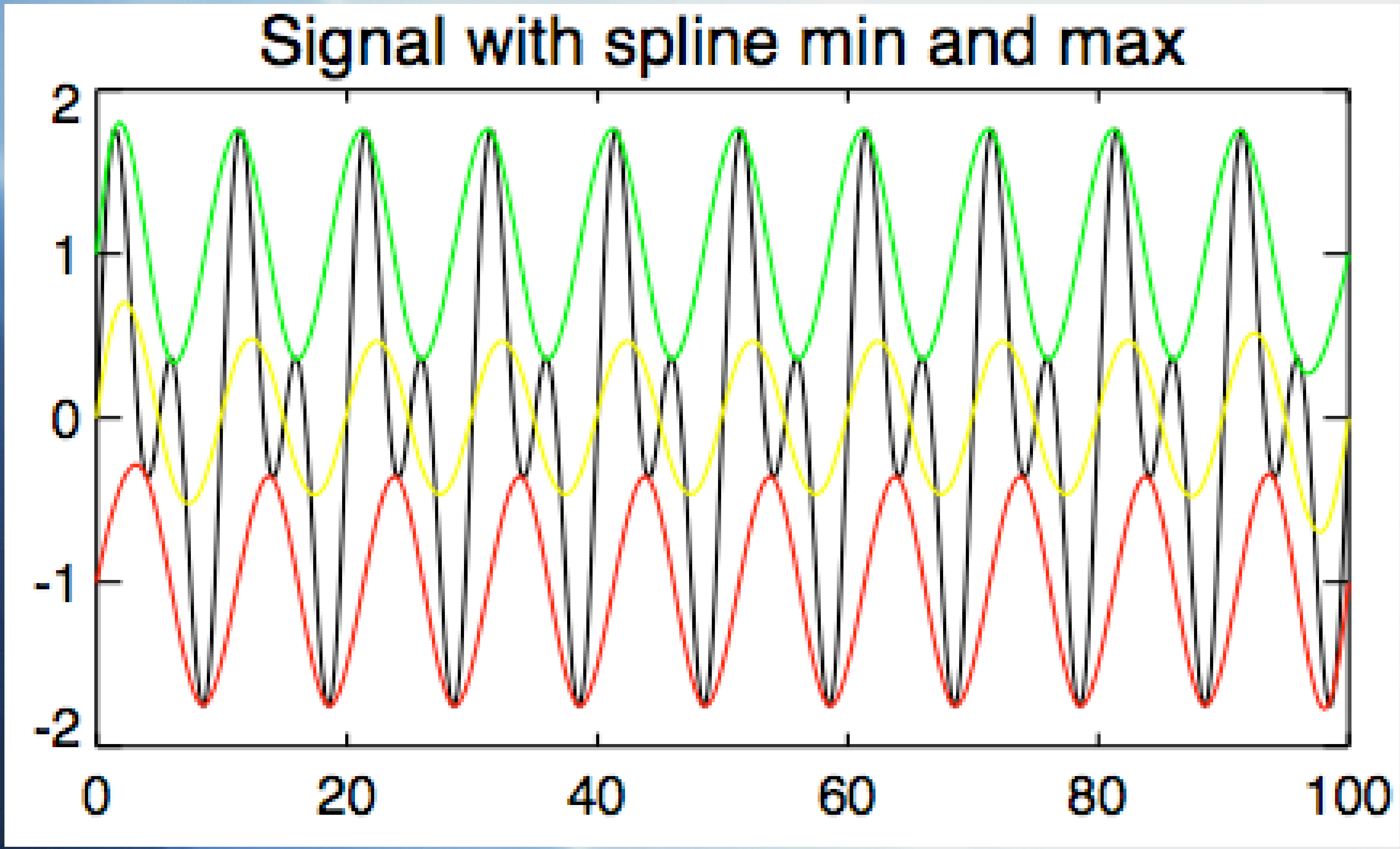
Problems With Analyzing Data

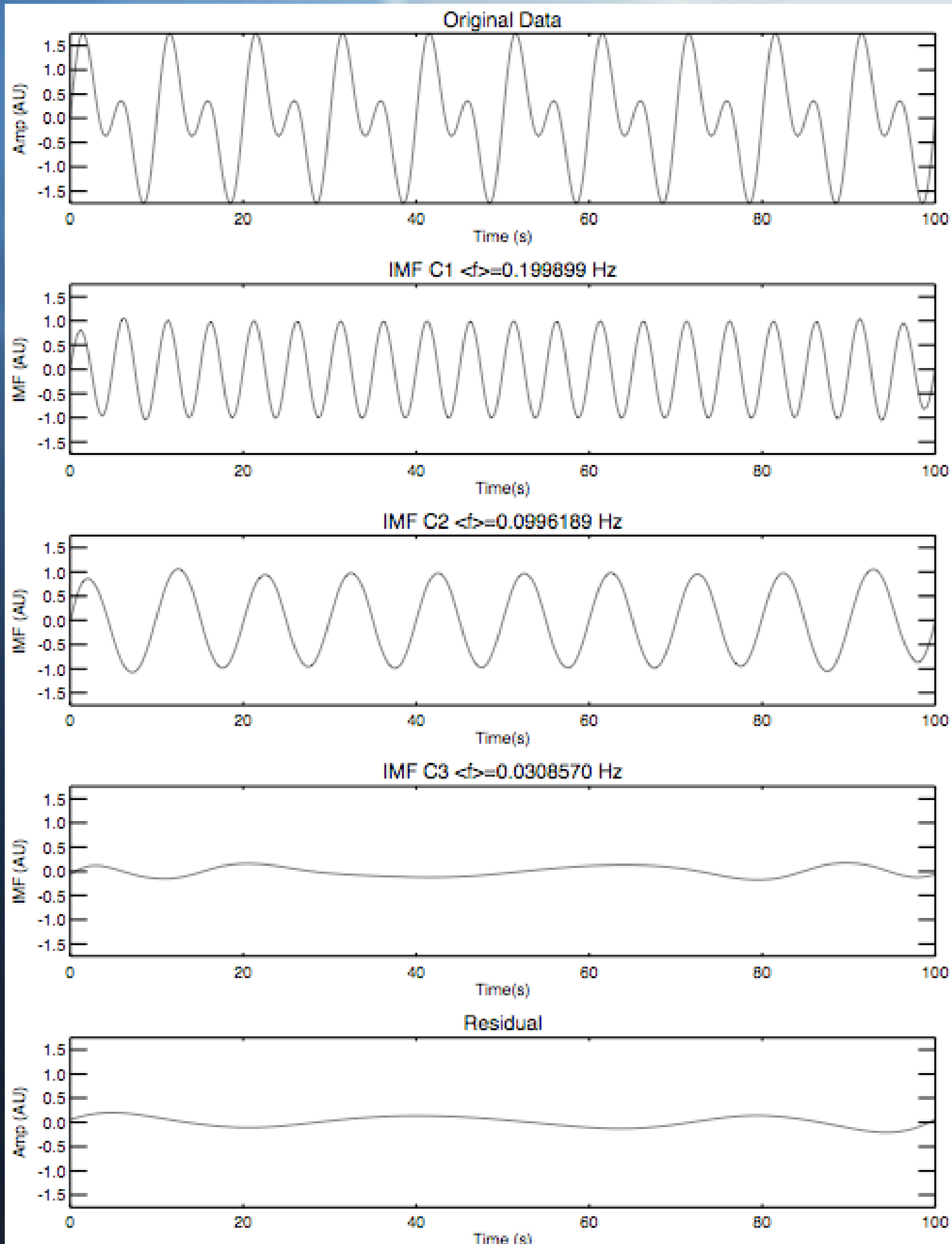
- Data is a time series, X(t).
- Most data is non-stationary.
- Some is nonlinear.
- Fourier Methods are of limited use.
  - Frequency of a signal changes within a characteristic period.
  - Spurious Harmonics to model nonlinear signal
- Hilbert Spectrum can solve the problem
  - Signal has to be conditioned to apply the Hilbert Transform. The signal must be symmetric with a zero mean.

Intrinsic Mode Functions (IMF)

- In order to apply a Hilbert Transform to a time series, X(t), the series must be symmetric about the local mean zero, with no positive minima or negative maxima.
- This is accomplished by 'sifting' a function using a cubic spline fit to the local minima and maxima.
- Once the splines have been fit, their mean is subtracted from the original time series. This process is repeated until the mean and standard deviation of the average spline is near zero. This is the first IMF, $C_1(t)$.
- The First IMF is subtracted from the original signal, and the residual is treated as a new time series to be sifted.

The Sifting Process





Simple model of two frequencies.

$Y(t)=\sin(2\pi f_1 t)+\sin(2\pi f_2 t)$ with f1=0.1Hz and f2=0.2 Hz.

The sifting process first isolates high frequencies, and the progresses to lower frequency modes.

The relevance or ‘quality’ of each IMF is determined by it’s correlation coefficient.

$$\gamma_i = \frac{\int X(t)C_i(t)dt}{\sqrt{\int X^2(t)dt \int C_i^2(t)dt}}$$

The Hilbert Transform

- The Hilbert Transform of a time series X(t) is given by

$$Y(t) = \frac{1}{\pi} PV \int \frac{X(t')}{t - t'} dt'$$

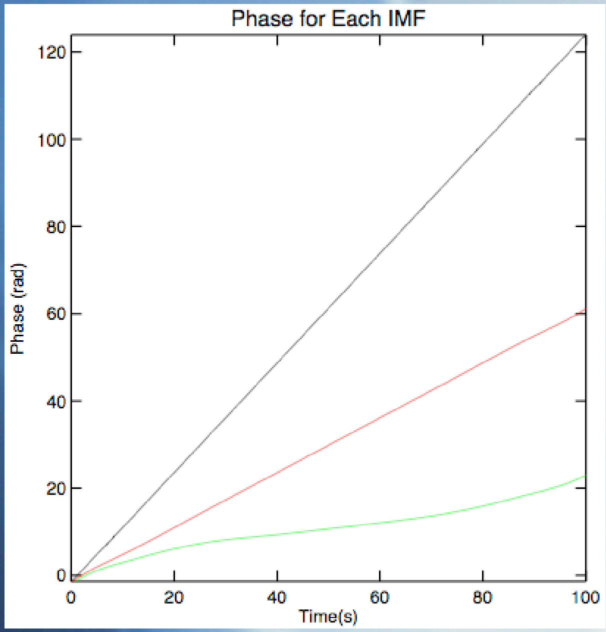
We can then form an analytic function Z(t).

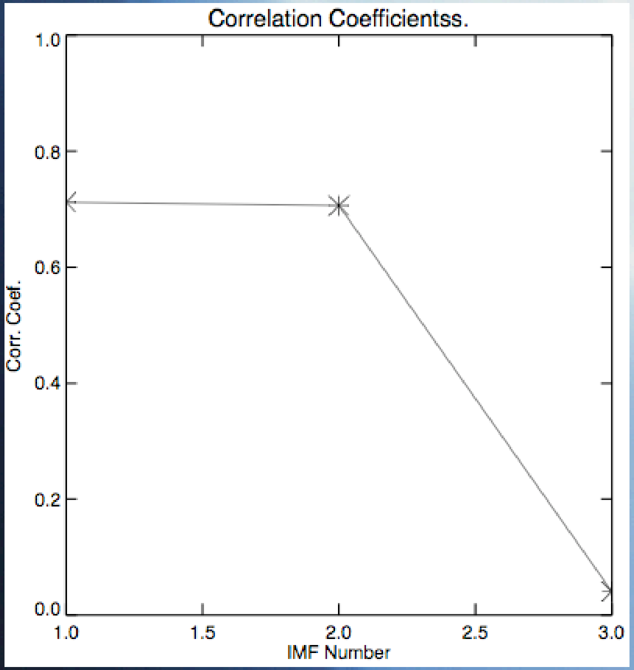
$$Z(t) = X(t) + iY(t) = a(t)e^{i\theta(t)}$$

The ‘Instantaneous Frequency’ is then obtained by

$$\omega(t) = \frac{d\theta(t)}{dt}$$

The phase must be ‘unwrapped’ before differentiating.

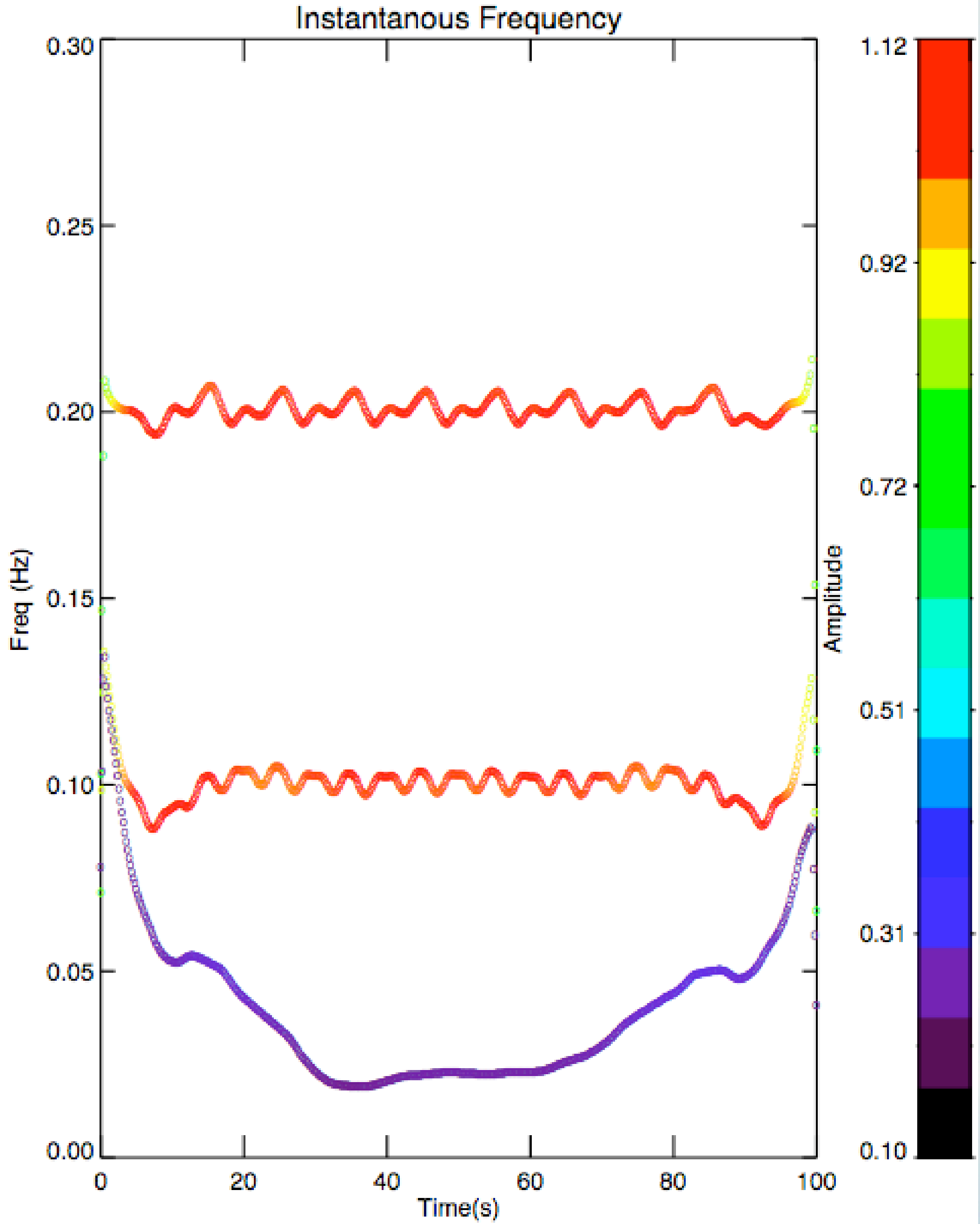




Here, we can see the unwrapped phase and correlation coefficient for the IMFs. The phases are approximately straight lines. The first two IMFs are significant, while the last is meaningless.

We can form a Hilbert Spectrum contour plot $H(t,\omega)$, which displays the amplitude(z-axis) as a function of time and frequency.

The spectrum H is qualitative, but certain integral quantities provide statistics.



The sifting process has isolated the two dominant modes in the system at the correct frequencies, 0.1, 0.2 Hz.

The third mode is insignificant in amplitude and correlation.

Hilbert Spectrum Statistics

Instantaneous Energy

$$IE(t) = \int_0^{\omega_N} H^2(t,\omega)\,d\omega$$

Mean Marginal Spectrum

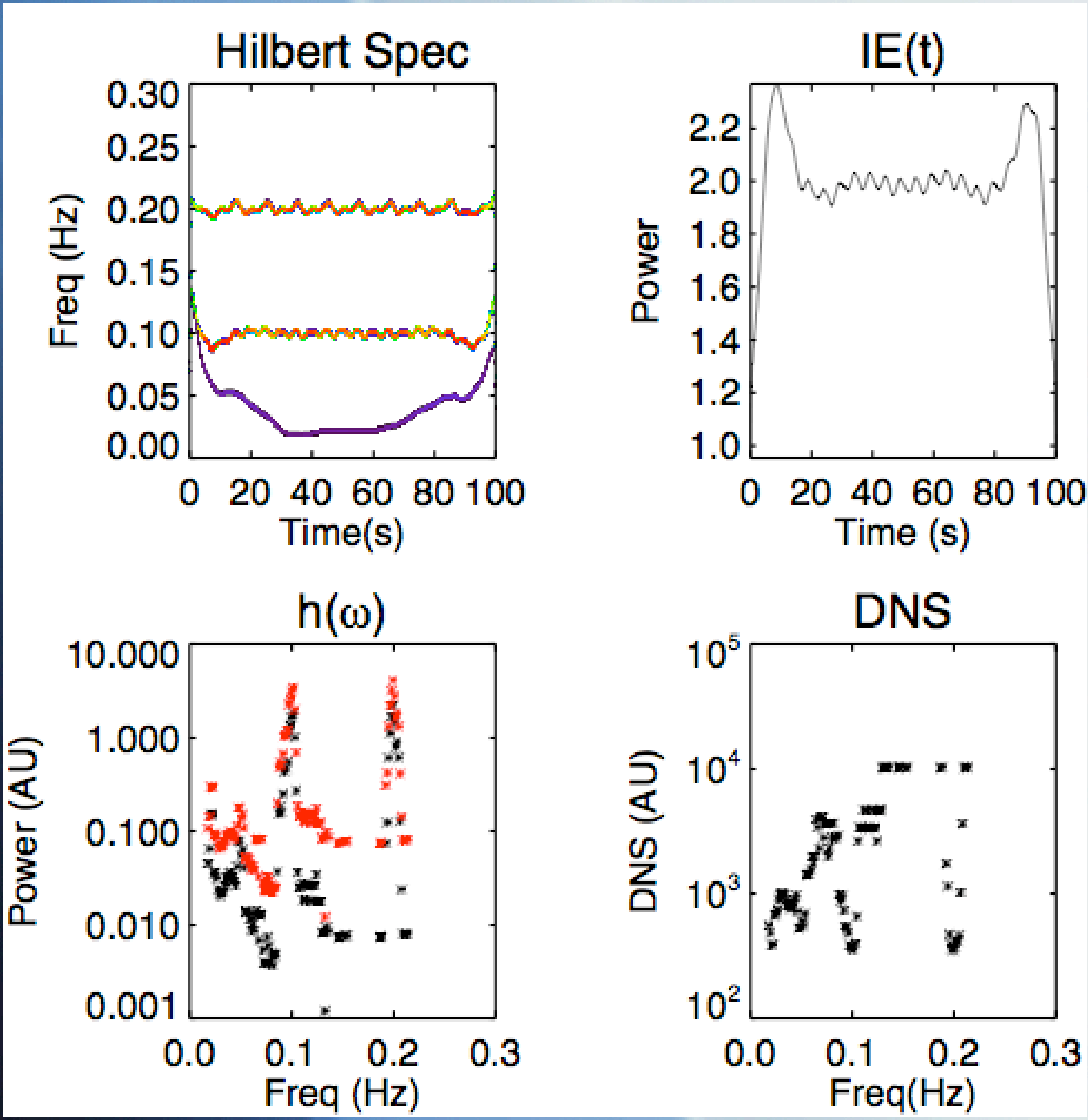
'Fourier-like' spectrum

$$h(\omega) = \frac{1}{T}\int_0^T H(t,\omega)\,dt$$

Degree of Non-Stationarity

A normalized measure of the deviation of the Hilbert Spectrum away from h(ω)

$$DNS(\omega) = \frac{1}{T}\int_0^T \left[1 - \frac{H(t,\omega)}{h(\omega)}\right]^2 dt$$



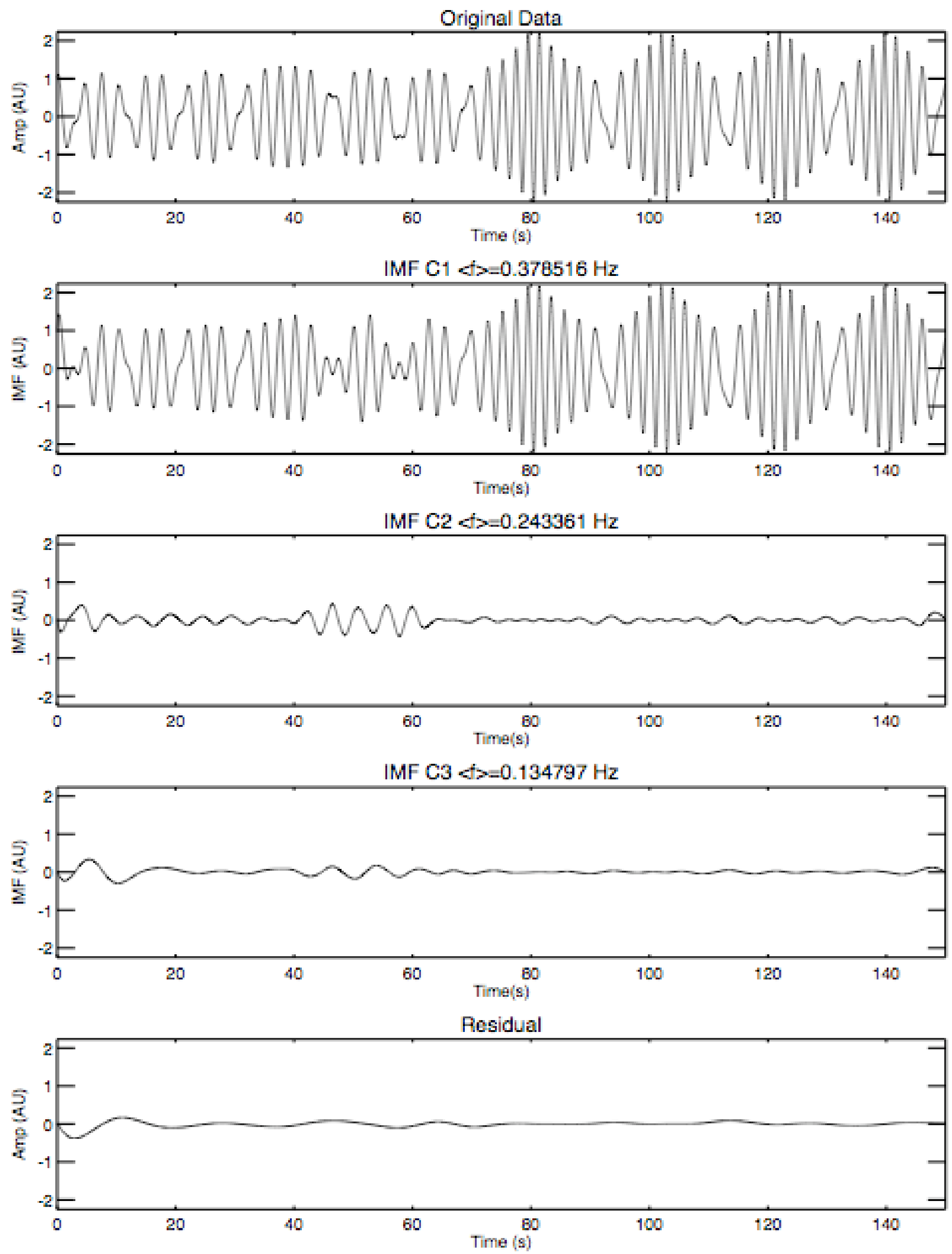
The spectrum has peaks at 0.1 and 0.2 Hz, as well as high stationarity.

Duffing's Equation

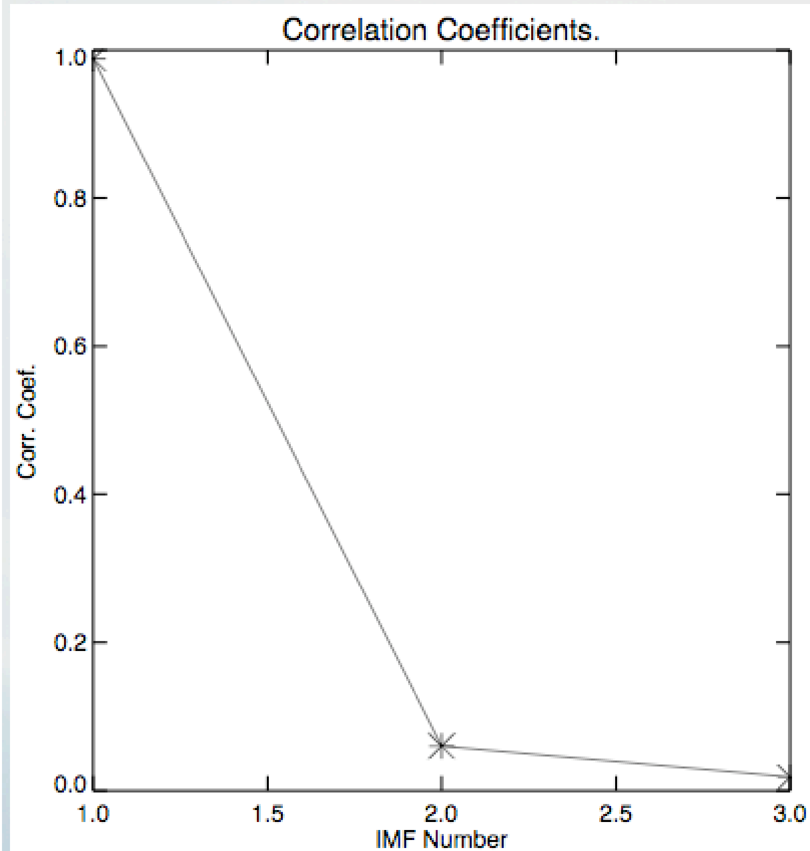
- Now, a classic nonlinear system.

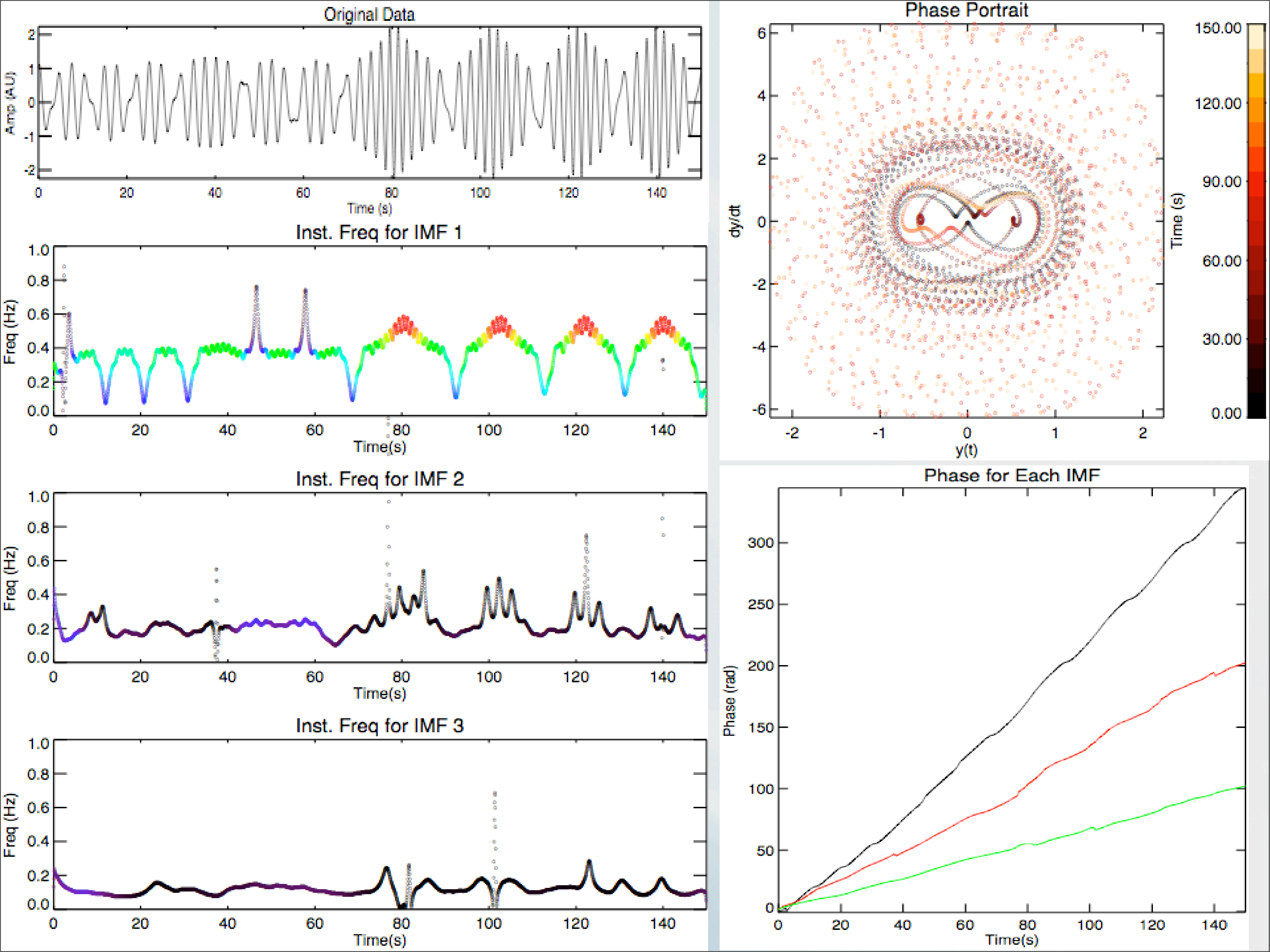
$$\frac{d^2x}{dt^2} + x(1 + \varepsilon x^2) = \gamma \cos(\omega t)$$

Forced harmonic oscillator with a nonlinear spring constant



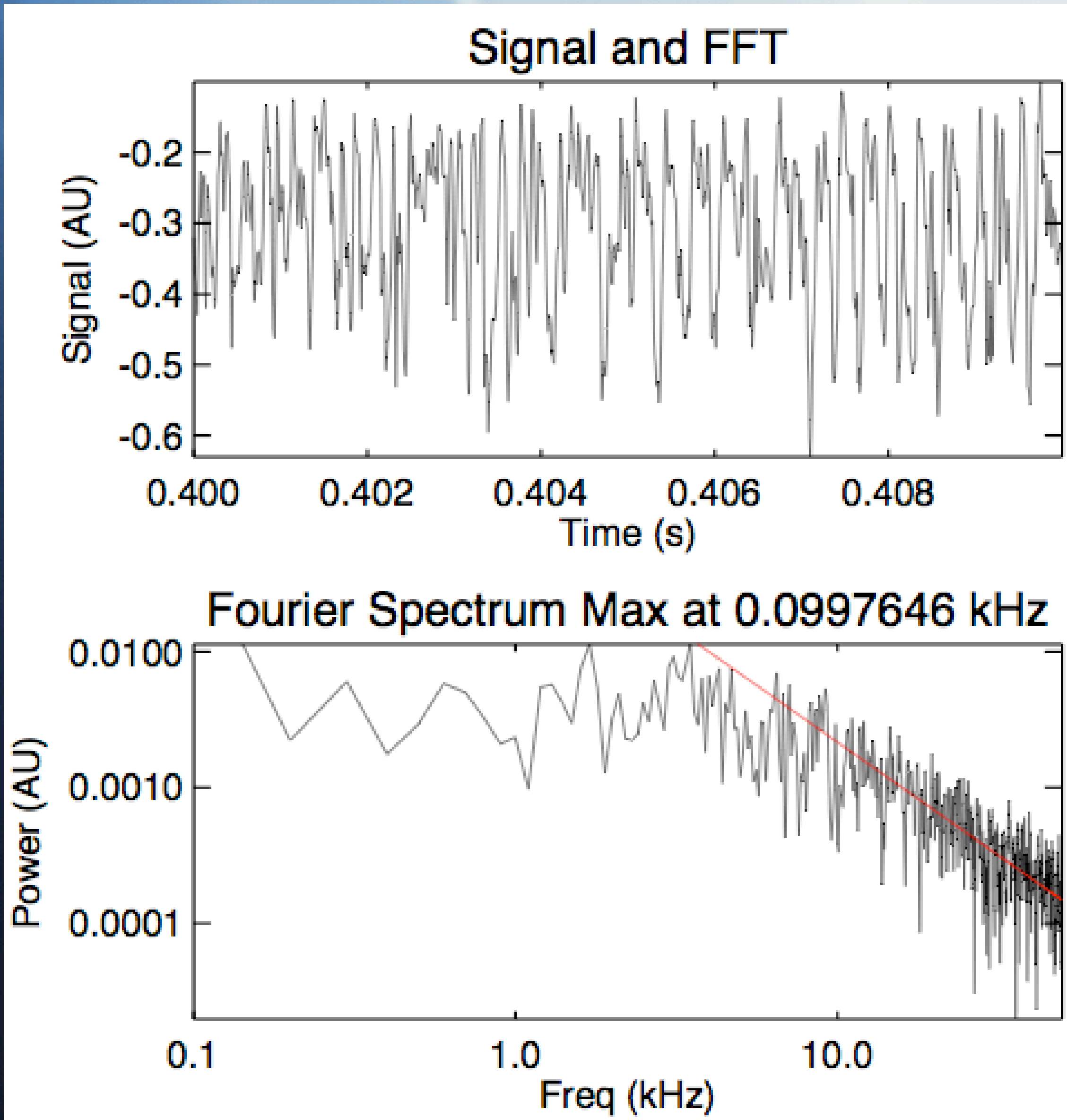
For Duffing's equation, the first IMF is most of the signal, except where there are local minima above zero, and local maxima below zero.





Original Data
Amp (AU)
Time (s)
Inst. Freq for IMF 1
Freq (Hz)
Time(s)
Inst. Freq for IMF 2
Freq (Hz)
Time(s)
Inst. Freq for IMF 3
Freq (Hz)
Time(s)
Phase Portrait
dy/dt
y(t)
Time (s)
Phase for Each IMF
Phase (rad)
Time(s)

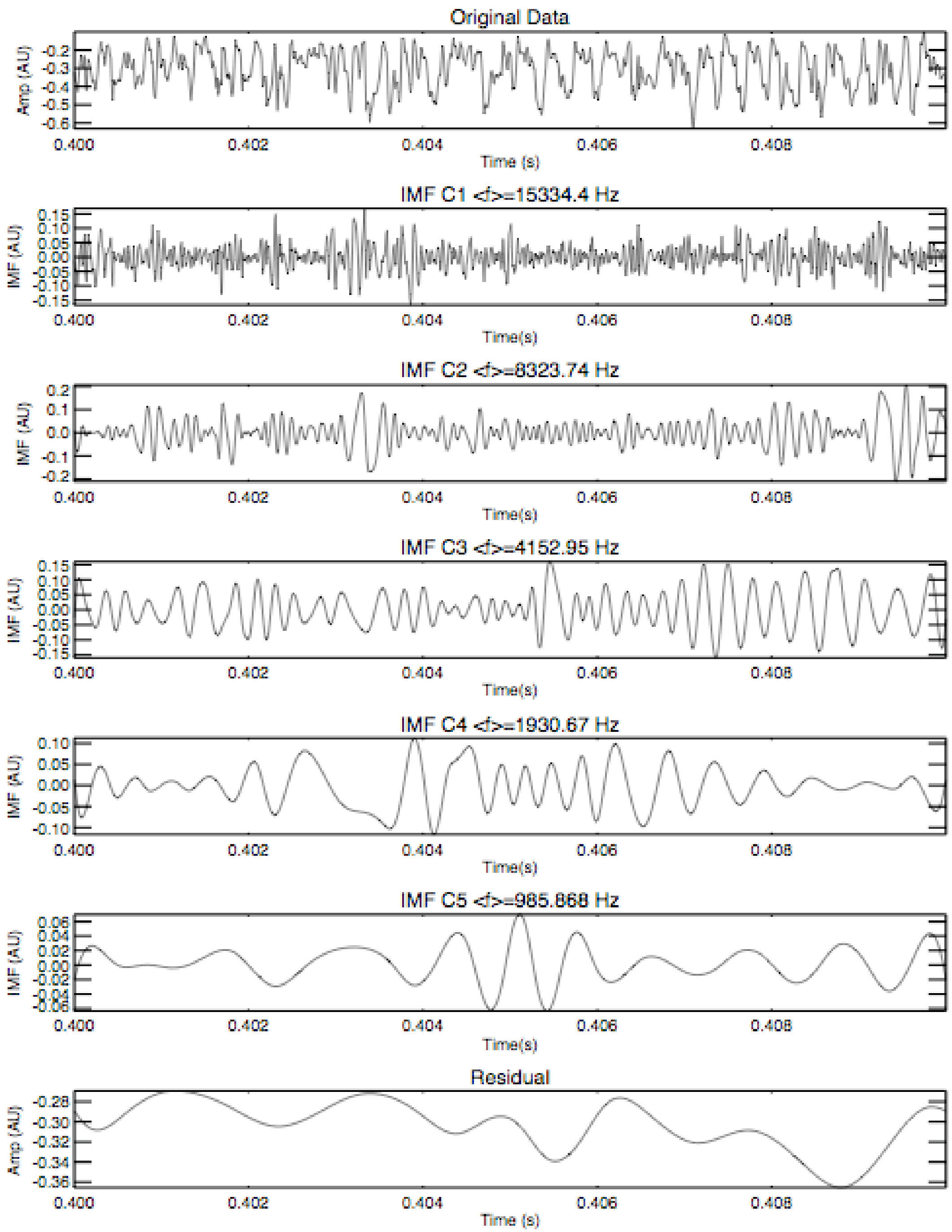
Turbulent Time Series



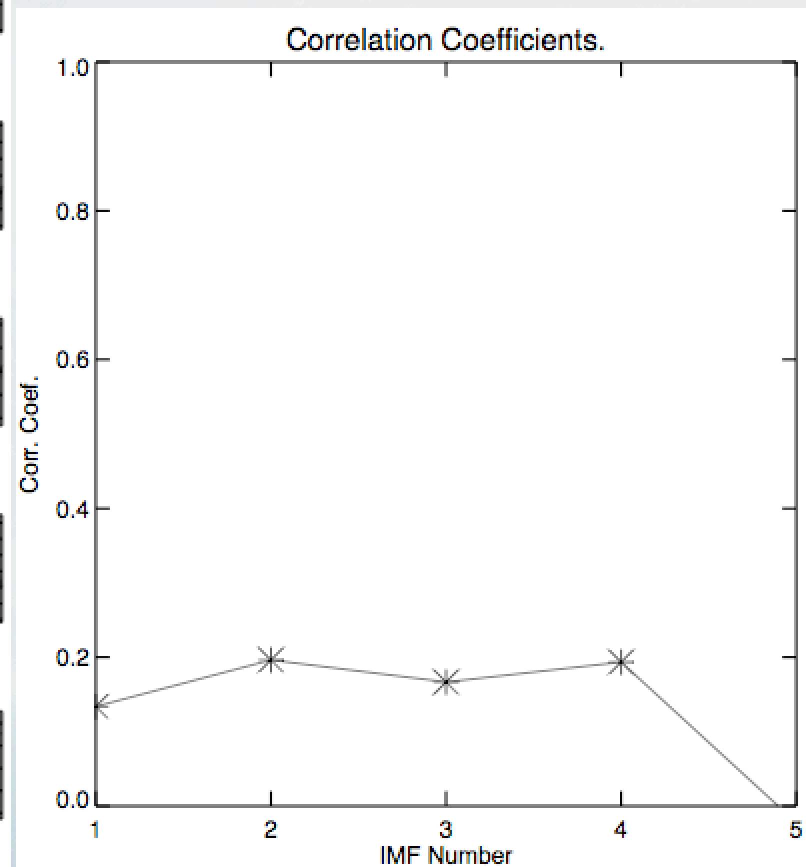
In the high density regime in CTX, the I_{sat} and floating potential data is non-stationary, with intermittent high frequency bursts. The FFT gives a power-law-like spectrum.

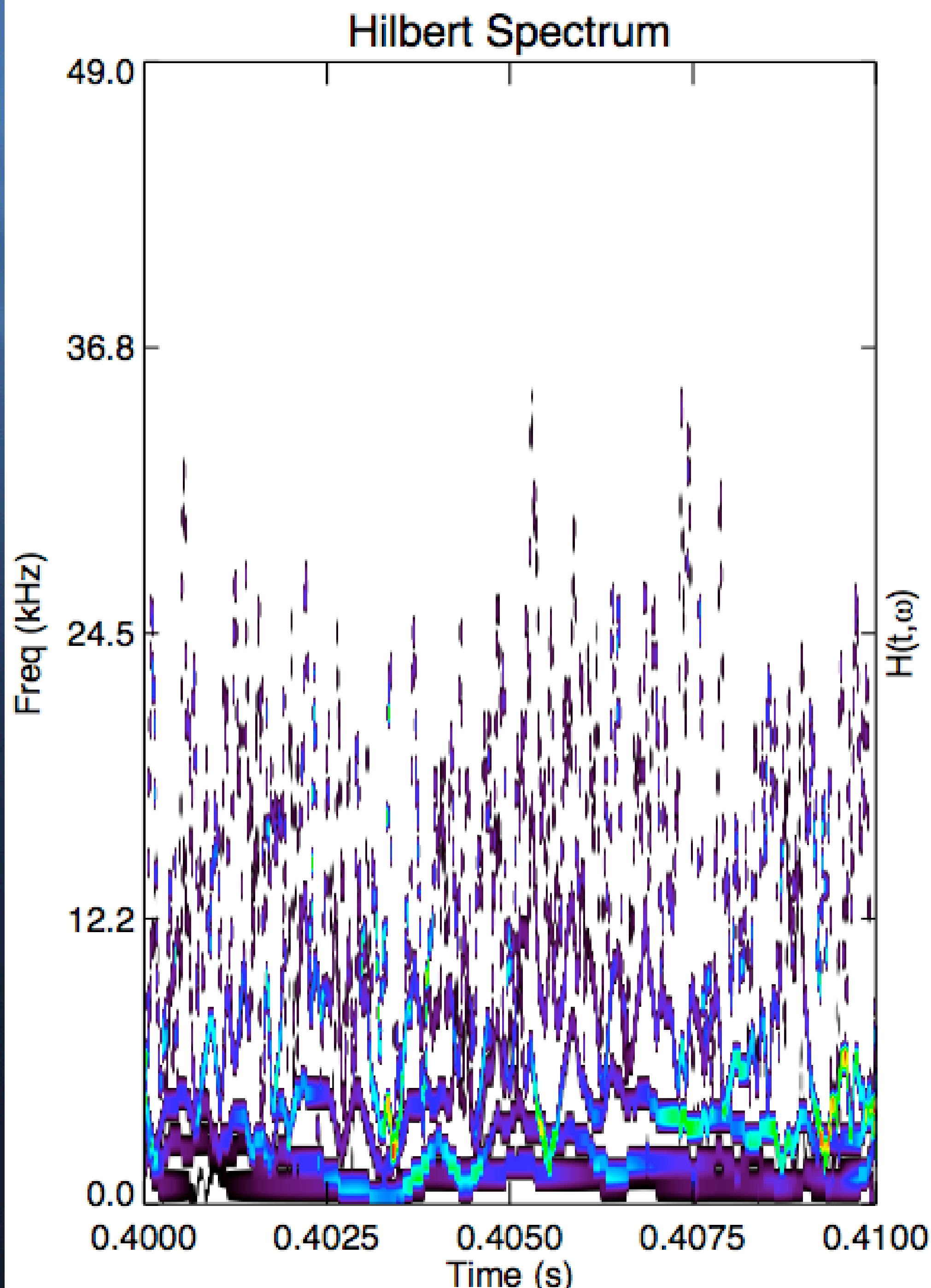
Shown in red is $P \sim f^{-5/3}$

Turbulent IMFs

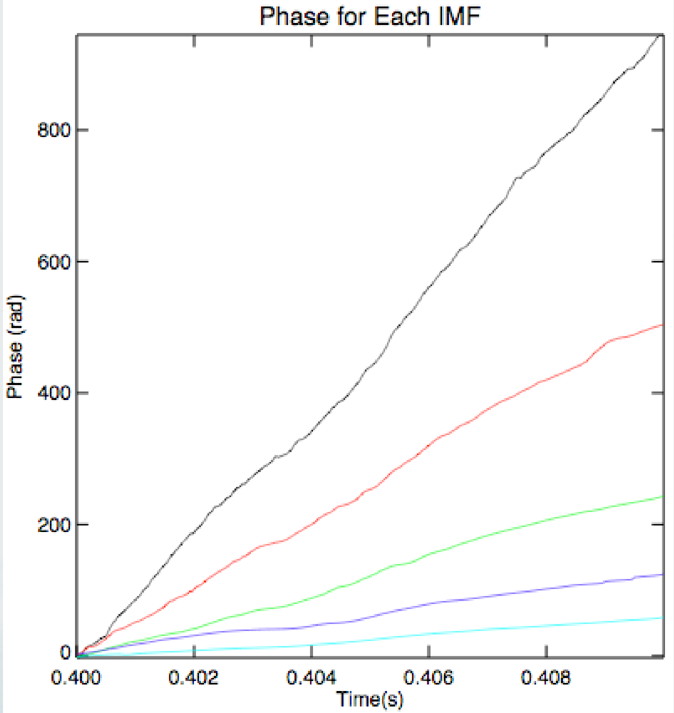


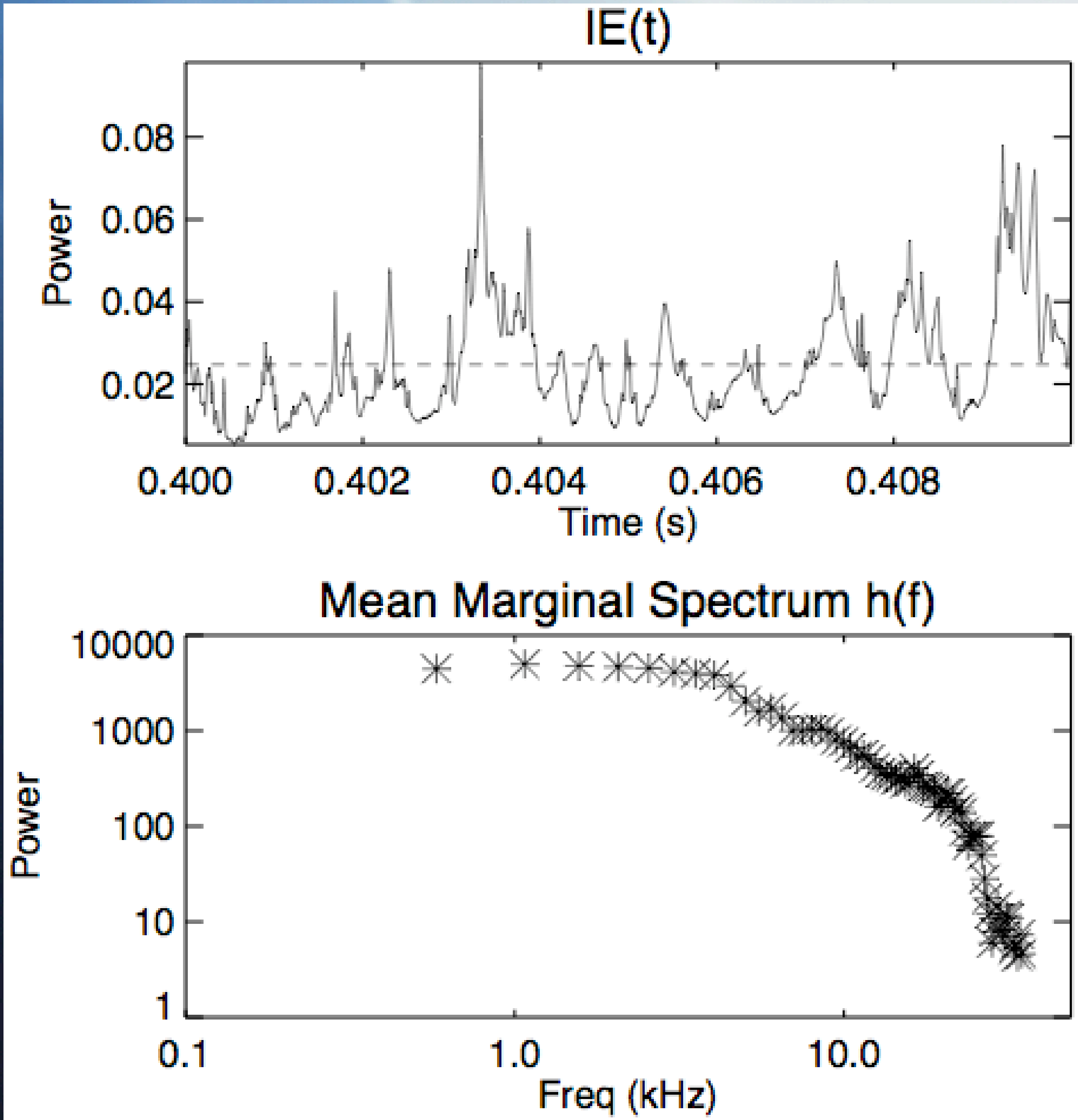
The turbulent time series can be decomposed into four simple modes!





The Hilbert spectrum displays intermittent high amplitude energetic modes, characteristic of turbulence.
The phases are well separated.

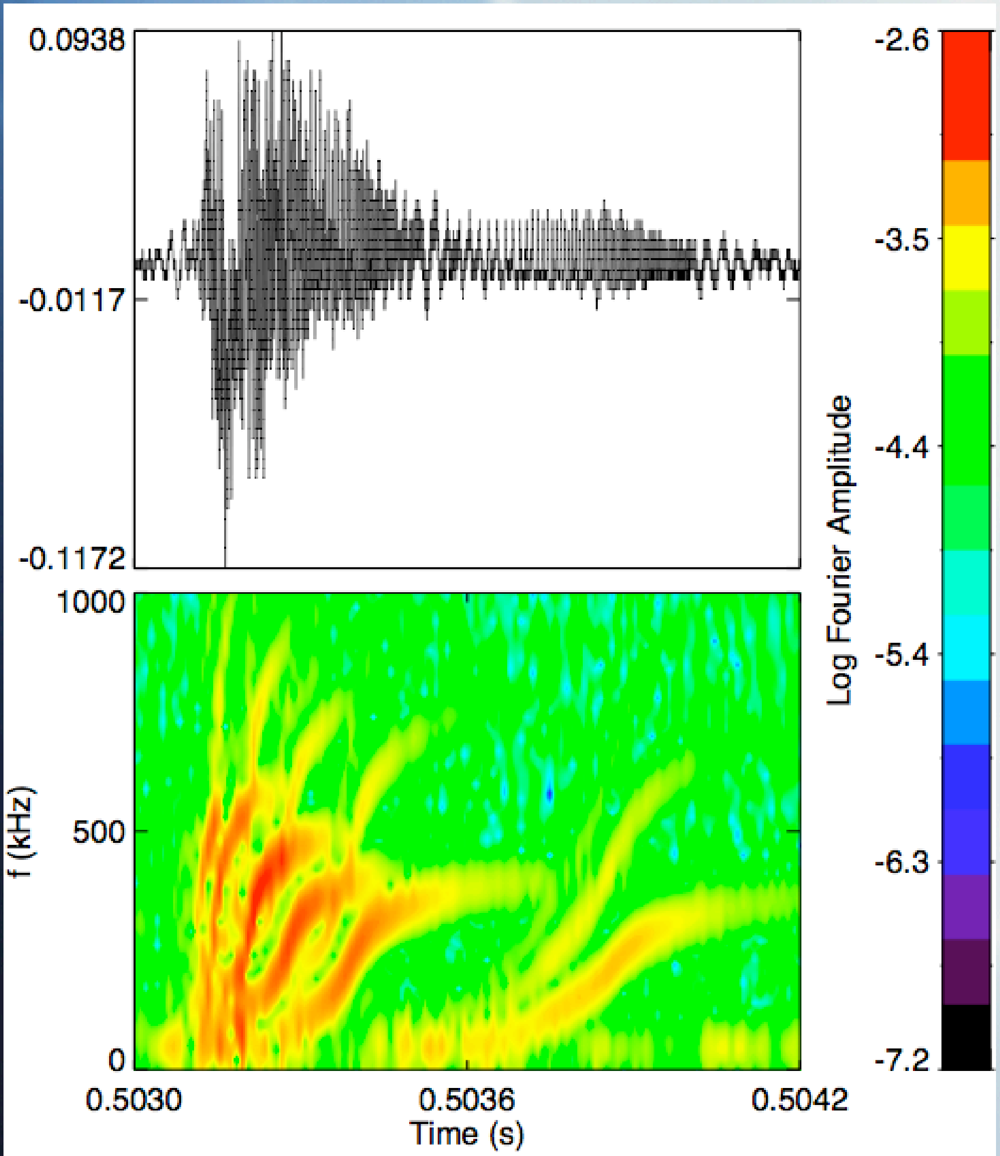




The instantaneous energy spectrum also shows the intermittent, high energy bursts of activity.

The spectrum is similar to the Fourier Spectrum, with a power law-like region.

IMFs and HEI

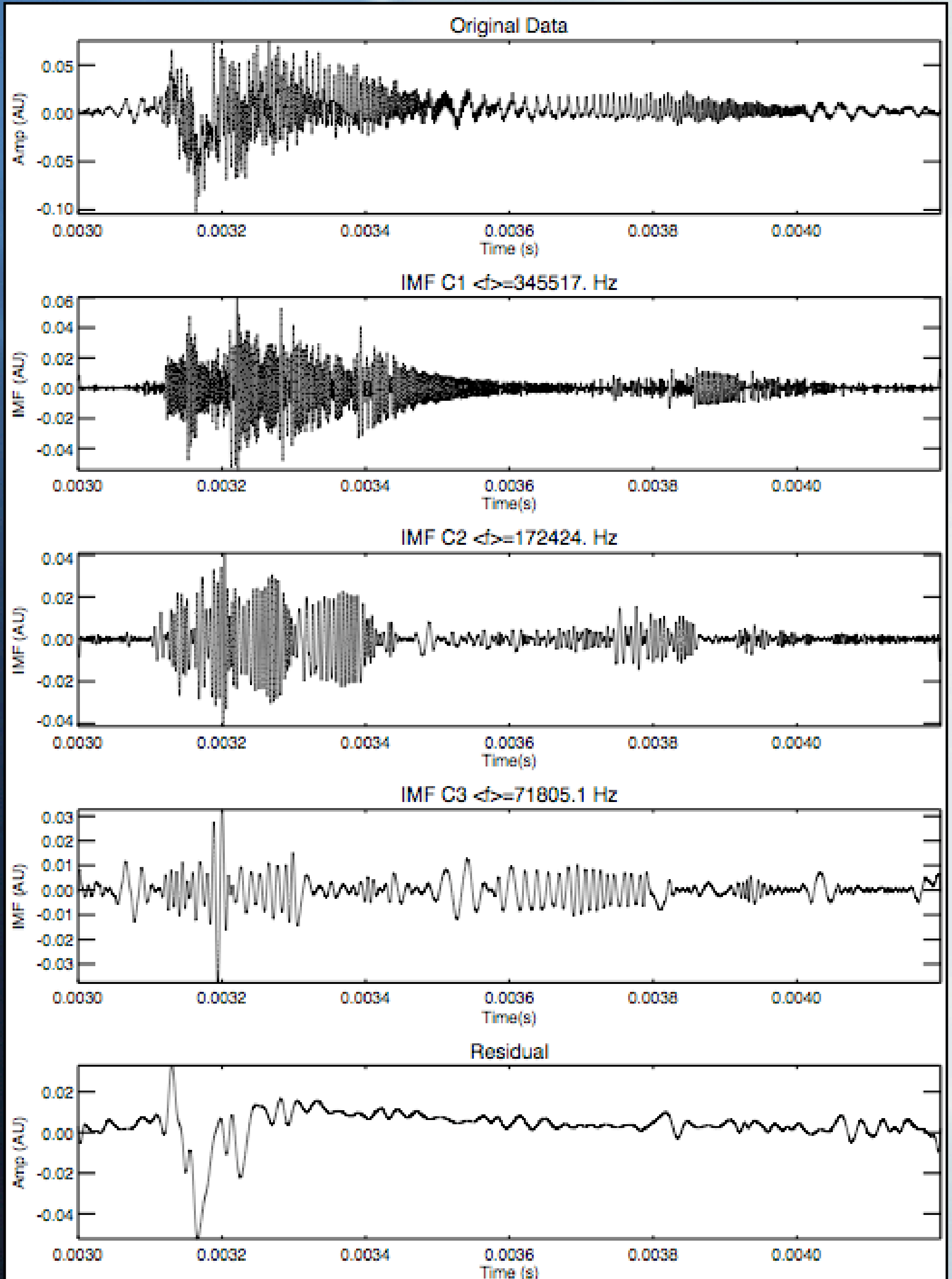


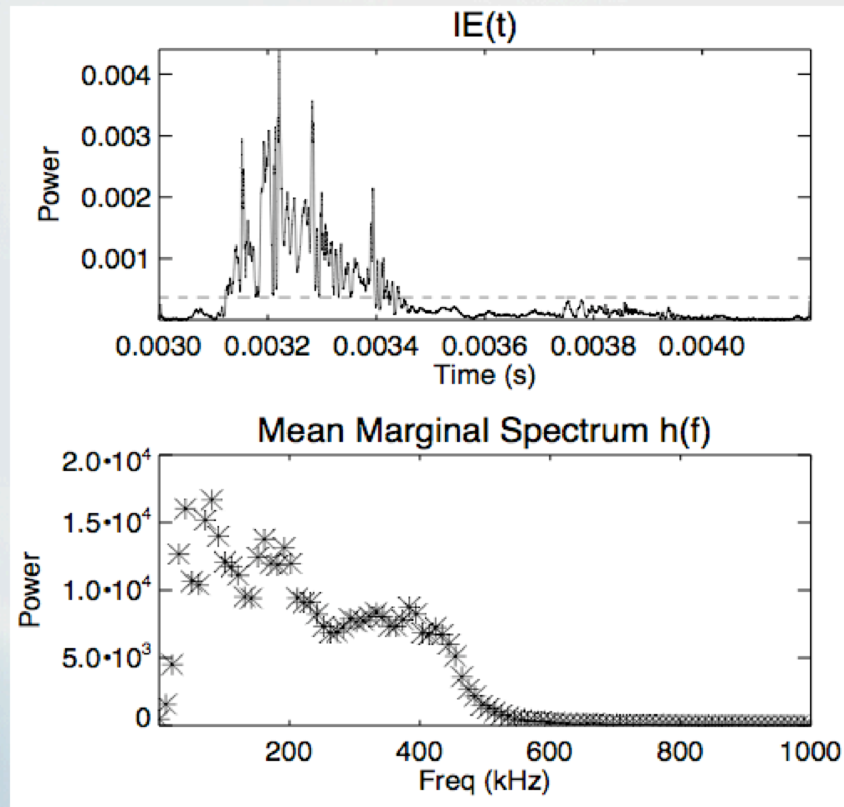
Using short-time Fourier Transforms, with an appropriate window, produces qualitative contours of frequency evolution during a HEI burst.

Using IMF decomposition and the Hilbert transform, we can extract mode growth and damping rates by examining the instantaneous amplitude in time.

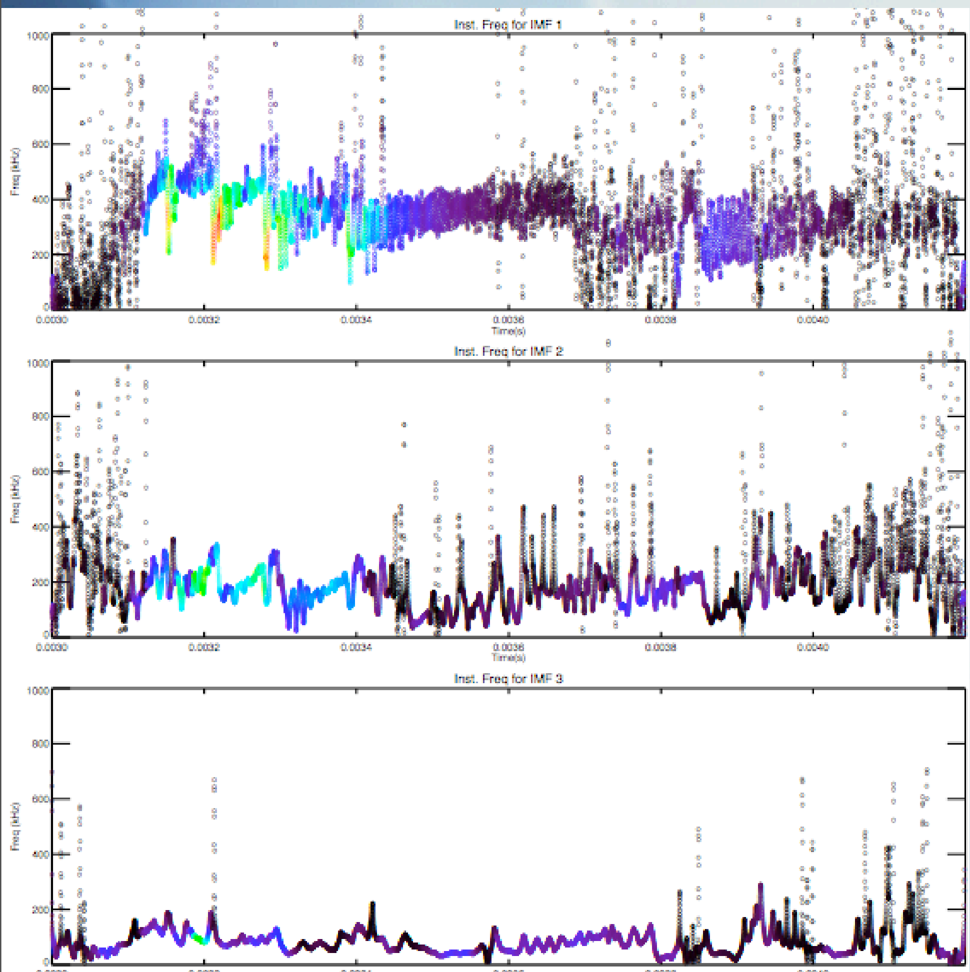
The first IMF $C_1(t)$ shows the damping of the high frequency, saturated modes.

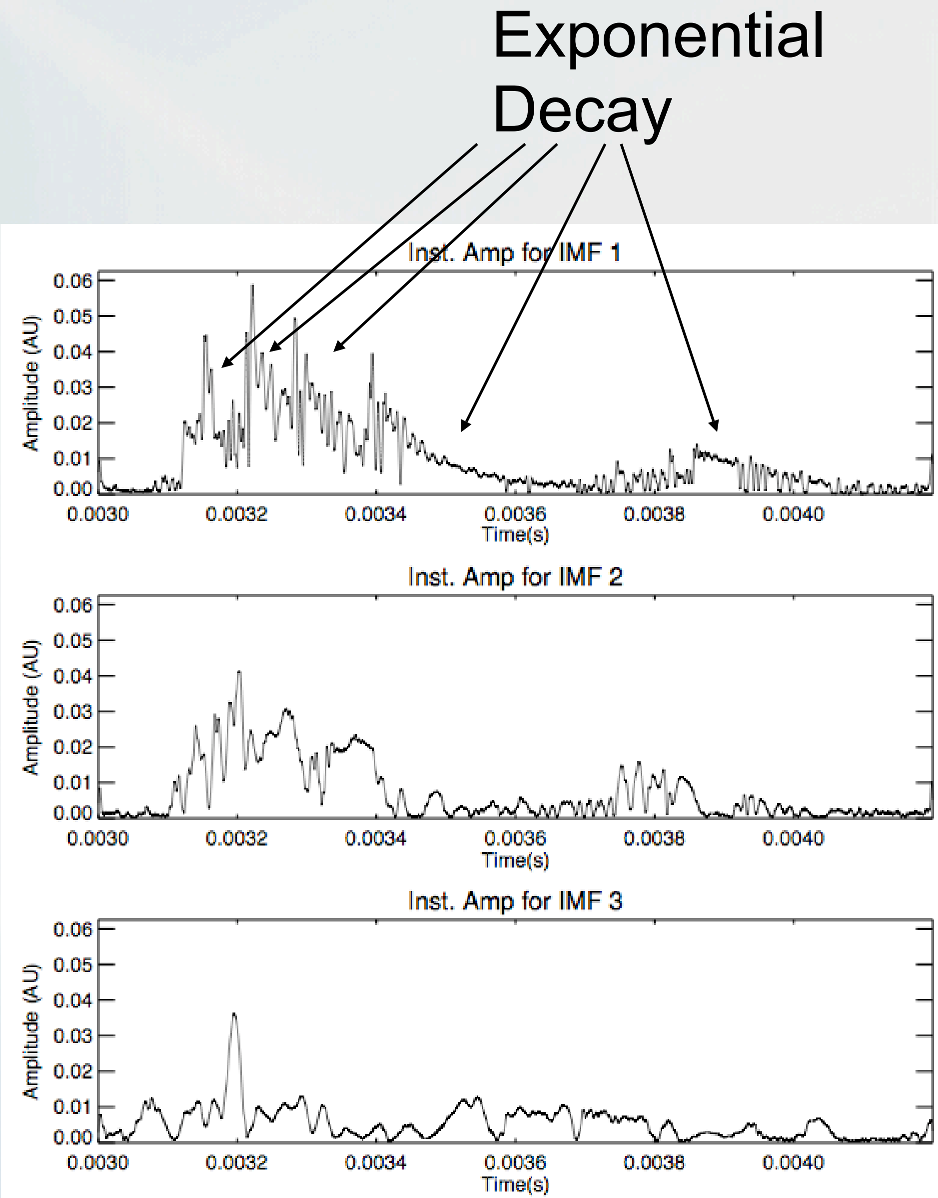
The second IMF shows the growth of those modes.





The Spectrum and Amplitude of the Individual IMFs allows extraction of mode damping rates! Here, $\gamma \sim -(6\text{-}12) \times 10^3\ s^{-1}$





Summary

- Decomposing one time series into multiple Intrinsic Mode Functions allows one to study the **Intrinsic Time Scales** of that time series.
- Once the modes have been separated, they can be individually studied with the Hilbert Transform to determine the relative importance of that time scale (power, frequency, etc…)
- The Hilbert Spectrum provides an instantaneous frequency and amplitude measurement, as well as important statistical quantities.