

# Equilibrium of an electron plasma confined on magnetic surfaces

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The difficulty in solving the equation for the electrostatic equilibrium of a pure electron plasma confined on magnetic surfaces can be reduced by separately solving for a part of the electric potential that is constant on the magnetic surfaces. Freedom exists in the choice for the constant part of the potential. Here a particular choice and how it simplifies the solution of the the electrostatic equilibrium problem are discussed. © 2005 American Institute of Physics.

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Although toroidal magnetic surfaces have been used for about half a century to confine quasineutral plasmas, proposals to confine non-neutral plasmas on magnetic surfaces are far more recent.<sup>1,2</sup> The theory of pure electron plasmas confined to magnetic surfaces differs significantly from that of quasineutral plasmas. For example, the equilibrium of quasineutral plasmas is primarily determined by solving force balance,  $\vec{\nabla}p = \vec{j} \times \vec{B}$  with  $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ . The electric field is determined by making the transport ambipolar, and the charge imbalance is given by  $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ . In pure electron plasmas, the plasma density is sufficiently low that its current has a negligible effect on the magnetic field. Force balance and rapid heat transport along the magnetic field lines determine<sup>2</sup> the form of the electron density,

$$n(\vec{x}) = N(\psi) \exp\left(\frac{e\Phi(\vec{x})}{T(\psi)}\right), \quad (1)$$

where  $\psi$  is the toroidal magnetic flux enclosed by a magnetic surface and  $T(\psi)$  is the electron temperature. The primary equilibrium equation is the equation for the electric potential

$$\nabla^2 \Phi = \frac{e}{\epsilon_0} n(\vec{x}) \quad (2)$$

with  $\vec{E} = -\vec{\nabla}\Phi$ . The theory of equilibrium for pure electron plasmas is developed in Refs. 2 and 3.

The equation for  $\Phi$  is not determined until two functions of  $\psi$  are specified,  $N(\psi)$  and  $T(\psi)$ . The specification of the function  $N(\psi)$  presents practical difficulties because it differs from the actual electron density  $n(\vec{x})$  by the exponential factor  $\exp(e\Phi/T)$ . The same exponential factor also makes the equation highly nonlinear and difficult to solve when  $e\Phi/T \gg 1$ . Both difficulties are ameliorated by first solving for a part of the electric potential that is constant on the magnetic surface  $\Phi_0(\psi)$ , and then finding the difference between the correct potential and the part that is constant on the surfaces,

$$\tilde{\Phi}(\vec{x}) \equiv \Phi(\vec{x}) - \Phi_0[\psi(\vec{x})]. \quad (3)$$

Much freedom exists in the choice of the potential  $\Phi_0(\psi)$  and in the associated part of the electron density  $\bar{\eta}(\psi)$ , which is typical electron density on a  $\psi$  surfaces. A choice of  $\Phi_0(\psi)$

and  $\bar{\eta}(\psi)$  will be given which simplifies the equilibrium problem.

Although neither  $\Phi_0(\psi)$  nor  $\bar{\eta}(\psi)$  is a surface average of the potential or the density, it is useful to define the average on a magnetic surface that encloses toroidal flux  $\psi$ . If  $(\psi, \theta, \varphi)$  are any well behaved coordinates and  $\mathcal{J}$  the coordinate Jacobian, an average over a magnetic surface of a function  $f(\vec{x})$  is defined by

$$\langle f \rangle \equiv \frac{\oint f \mathcal{J} d\theta d\varphi}{\oint \mathcal{J} d\theta d\varphi}. \quad (4)$$

$V'(\psi) = \oint \mathcal{J} d\theta d\varphi$  is the  $\psi$  derivative of the volume enclosed by a magnetic surface.

To have a well-defined set of equations (1) the parts of the potential and the electron density that are constant on the  $\psi$  surfaces,  $\Phi_0$  and  $\bar{\eta}$ , must be related to each other and (2) the typical, or constant part of the, electron density  $\bar{\eta}$  must be related to the actual electron density  $n(\vec{x})$ .

A natural relation between the part of the electric potential that is constant on the magnetic surfaces  $\Phi_0(\psi)$  and the typical electron density on a  $\psi$  surface  $\bar{\eta}(\psi)$  is

$$\langle \nabla^2 \Phi_0 \rangle = \frac{e}{\epsilon_0} \bar{\eta}(\psi), \quad (5)$$

where

$$\langle \nabla^2 \Phi_0 \rangle = \frac{1}{V'} \frac{d}{d\psi} \left( V' \langle g^{\psi\psi} \rangle \frac{d\Phi_0}{d\psi} \right), \quad (6)$$

where  $g^{\psi\psi} \equiv \vec{\nabla}\psi \cdot \vec{\nabla}\psi$  is a component of the metric tensor.

A natural relation between the typical electron density  $\bar{\eta}(\psi)$  and the actual electron density  $n(\vec{x})$  is

$$n(\vec{x}) = \bar{\eta}(\psi) \exp\left(\frac{e\tilde{\Phi}}{T}\right). \quad (7)$$

This equation is equivalent to defining the typical electron density to be  $\bar{\eta}(\psi) \equiv N(\psi) \exp(e\Phi_0/T)$ . Since  $\bar{\eta}$  is a typical density on a  $\psi$  surface, it is advantageous to use  $\bar{\eta}(\psi)$ , instead of  $N(\psi)$ , to specify the density.

The solution to the differential equation for  $\Phi_0(\psi)$  can be given as an integral. The effective charge enclosed by a  $\psi$  surface is

$$\mathcal{Q}(\psi) \equiv -e \int_0^\psi \bar{\eta}(\psi) V' d\psi, \quad (8)$$

though  $\mathcal{Q}(\psi)$  is generally not the actual charge enclosed by the surface. The constant part of the potential is

$$\Phi_0(\psi) = \Phi_0(0) - \frac{1}{\epsilon_0} \int_0^\psi \frac{\mathcal{Q}(\psi)}{V' \langle g^{\psi/\psi} \rangle} d\psi, \quad (9)$$

where  $\Phi_0(0)$  is the value of the constant part of the electric potential on the magnetic axis, the curve on which  $\psi=0$ . Near the magnetic axis  $\psi \rightarrow 0$ , analyticity implies  $V'$  constant,  $\langle g^{\psi/\psi} \rangle \propto \psi$ , and  $\mathcal{Q} \propto \psi$ .

Using the natural relations as the definitions for the constant part of the potential  $\Phi_0(\psi)$  and the typical density  $\bar{\eta}(\psi)$ , the equation for the variable part of the potential  $\tilde{\Phi}(\vec{x})$  is obtained from the requirement that  $\Phi = \Phi_0 + \tilde{\Phi}$  obeys the equation  $\nabla^2 \Phi = en(\vec{x})/\epsilon_0$ . Letting

$$\eta(\vec{x}) \equiv \frac{\epsilon_0}{e} \nabla^2 \Phi_0(\psi), \quad (10)$$

which implies  $\bar{\eta}(\psi) = \langle \eta(\vec{x}) \rangle$ , one finds that

$$\frac{\epsilon_0}{e} \nabla^2 \tilde{\Phi} - \bar{\eta}(\psi) e^{e\tilde{\Phi}/T} = -\eta(\vec{x}). \quad (11)$$

This equation is the minimum of the energy

$$W = \int \left\{ \frac{\epsilon_0}{2} (\vec{\nabla} \tilde{\Phi})^2 + \bar{\eta} T \exp\left(\frac{e\tilde{\Phi}}{T}\right) - \eta e \tilde{\Phi} \right\} d^3x. \quad (12)$$

The first two terms of the energy integrand are always positive since the typical electron density  $\bar{\eta}(\psi)$  is positive. The last term in the integrand can be of either sign because both  $\eta(\vec{x})$  and  $\tilde{\Phi}$  can have either sign.

Equation (11) can be solved directly, though it can be simplified in a number of limits. For example, Eq. (11) is easily solved when the electron density is sufficiently low so that the Debye length,  $\lambda_d^2(\psi) \equiv T\epsilon_0/(e^2 \bar{\eta})$ , is long compared to the spatial gradient scale  $a_g$  or more precisely if  $\lambda_d^2 \times \exp(-e\tilde{\Phi}/T) \gg a_g^2$ . In this limit the solution to Poisson's equation  $\epsilon_0 \nabla^2 \Phi = en$  differs by a small amount compared to  $T/e$  from the solution to Laplace's equation  $\epsilon_0 \nabla^2 \Phi = 0$ .

Laplace's equation is equivalent to  $(\epsilon_0/e) \nabla^2 \tilde{\Phi} = -\eta(\vec{x})$ . Pedersen<sup>3</sup> has noted that when the Debye length is long and the plasma is surrounded by a good conductor,  $|e\tilde{\Phi}/T| \ll 1$ . Equation (11) can then be approximated as  $(\epsilon_0/e) \nabla^2 \tilde{\Phi} = -\eta(\vec{x}) + \bar{\eta}[\psi(\vec{x})]$ .

When  $e\tilde{\Phi}/T$  is positive and sufficiently large for the term  $\bar{\eta} T \exp(e\tilde{\Phi}/T)$  to be large, the minimum of the energy  $W$  is given by a balance of the last two terms, which means  $\exp(e\tilde{\Phi}/T) \approx \eta/\bar{\eta}$ , where  $\eta$  is positive.

In locations where  $\eta$  is negative, the spatial gradients must be sufficiently sharp for the  $\nabla^2 \tilde{\Phi}$  term in Eq. (11) to provide a balance, which implies the electron density can vary rapidly along the magnetic field lines in regions where  $\eta < 0$ .

Equation (11) can be solved by using one of the limiting cases that has been discussed except in boundary layers that have a thickness of order a Debye length  $\lambda_d$ . In general, obtaining a solution to Eq. (11) for  $\tilde{\Phi}(\vec{x})$  should be of comparable difficulty to solving the equilibrium equation in the limit where the Debye length is comparable or longer than the size of the plasma, which is the easier limit.

The determination of a part of the electric potential that is constant on the magnetic surfaces produces three simplifications. First, the typical electron number density on a magnetic surface  $\bar{\eta}(\psi)$  is a more natural quantity for specifying equilibria than is  $N(\psi)$ , which can differ by many orders of magnitude from the electron density at all points on the  $\psi$  surface. Second, qualitative features of equilibria are more easily derived—for example, the qualitatively different behavior of equilibria in regions where  $\eta \equiv -(\epsilon_0/e) \nabla^2 \Phi_0(\psi)$  is positive and negative. Third, numerical solutions for equilibria should be simpler when the electron density is sufficiently large so that the Debye length is much shorter than the plasma radius.

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<sup>2</sup>T. S. Pedersen and A. H. Boozer, Phys. Rev. Lett. **88**, 205002 (2002).

<sup>3</sup>T. S. Pedersen, Phys. Plasmas **10**, 334 (2003).