Density limit for electron plasmas confined by magnetic surfaces

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The density limit for electron plasmas confined by toroidal magnetic surfaces is investigated. In a cylinder, the well-known limit is the Brillouin density, $n_B \equiv \epsilon_0 B^2/2m_e$. In an axisymmetric torus, the confining region shifts outward in major radius, and this shift is shown to equal half the plasma radius when $n/n_B \approx \iota^2 a/R_0$, where $\iota = 1/q$ is the rotational transform of the magnetic field and a/R_0 is the inverse aspect ratio of the torus. In a nonaxisymmetric torus, electron confinement is found to be lost due to stochasticity effects when $n/n_B \approx (\iota^2/8M^2)(a/R_0)^2/\delta_{MN}$. The asymmetry amplitudes δ_{MN} are the fractional variations in n/B^2 on a magnetic surface in the poloidal mode number M and the toroidal mode number $N \approx \iota M$. *© 2005 American Institute of Physics.*
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As is well known, but also shown below, the density of a pure electron plasma that is confined by a magnetic field is limited by the Brillouin density,[1]

$$n_B \equiv \frac{\epsilon_0 B^2}{2m_e}. \tag{1}$$

When the electrons are confined by toroidal magnetic surfaces,[2,3] it will be shown that density limit can be much lower than the Brillouin limit. The reduction can be especially large when the surfaces are toroidally asymmetric as in a stellarator.

A density limit occurs when the inertial forces associated with the $\mathbf{E}\times\mathbf{B}$ flow of the electrons become sufficiently strong compared to the $\boldsymbol{v}\times\mathbf{B}$ force. If the electron temperature is a spatial constant, the electron force-balance equation,

$$\frac{m_e}{e}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\nabla p}{en} = \nabla\Phi - \boldsymbol{v}\times\mathbf{B}, \tag{2}$$

can be rewritten as

$$\nabla\Phi_* = \boldsymbol{v}\times\mathbf{B}_*. \tag{3}$$

Both the effective electric potential,

$$\Phi_* \equiv \Phi - \frac{m_e}{2e}v^2 - \frac{T}{e}\ln(n), \tag{4}$$

and the effective magnetic field,

$$\mathbf{B}_* \equiv \mathbf{B} - \frac{m_e}{e}\nabla\times\boldsymbol{v}, \tag{5}$$

depend on the electron inertia. It is the inertial effect on the effective magnetic field that reduces the limit on the density to a lower value than that found by Brillouin. The relation between electron density and the electric potential is given by Poisson's equation,

$$\nabla^2\Phi = \frac{e}{\epsilon_0}n. \tag{6}$$

The electron density is so low in a pure electron plasma that it can carry insufficient current to modify the magnetic field. Consequently, $\nabla\times\mathbf{B}=0$.

The Brillouin density limit is obtained for zero pressure, $p=nT=0$, electrons with constant density that are confined in a cylinder in a magnetic field $\mathbf{B}=B\hat{z}$. Poisson's equation implies $\Phi=(en/4\epsilon_0)r^2$. Writing $\boldsymbol{v}=r\omega(r)\hat{\theta}$, one finds Eq. (3) can be written as $\omega^2-(eB/m_e)\omega+e^2n/2m_e\epsilon_0$, which has no solution if $n>n_B$. The intuitive explanation for the Brillouin limit is that the electrostatic repulsion, the electron pressure, and the centrifugal force $nm_e v^2/r$ are all radially outward; only the $en\boldsymbol{v}\times\mathbf{B}$ force is radially inward. Once the velocity v is sufficiently large that the centrifugal force is comparable to $envB$, a flow velocity v can no longer be found that gives force balance.

If an electron plasma is confined by magnetic surfaces then these surfaces must be toroidal and the magnetic field can be represented as[4]

$$2\pi\mathbf{B} = \nabla\psi\times\nabla\theta + \iota(\psi)\nabla\varphi\times\nabla\psi. \tag{7}$$

The magnetic surfaces are constant ψ surfaces since $\mathbf{B}\cdot\nabla\psi(\mathbf{x})=0$, where ψ is the toroidal magnetic flux enclosed by a magnetic surface, θ is a poloidal angle, and φ is a toroidal angle. The rotational transform ι is the inverse of the tokamak safety factor, $q(\psi)=1/\iota$. If $\iota=N/M$, the magnetic-field lines close on themselves after M toroidal circuits and N poloidal circuits. Such ψ surfaces are called rational magnetic surfaces.

Both the effective magnetic field $\mathbf{B}_*$ and the electron fluid velocity $\boldsymbol{v}$ lie in the constant Φ_* surfaces. This follows from Eq. (3), which implies that $\mathbf{B}_*\cdot\nabla\Phi_*=0$ and that $\boldsymbol{v}\cdot\nabla\Phi_*=0$. Consequently, the electron flow remains confined in a toroidal region only if the field lines of $\mathbf{B}_*$ remain confined. We will find that the loss of $\mathbf{B}_*$ magnetic surfaces can provide a more stringent limit on the electron density than the Brillouin limit.

If the density limit set by the loss of the $\mathbf{B}_*$ magnetic surfaces is far below the Brillouin limit, then the field lines of the $\mathbf{B}_*$ field can be found by perturbation theory. Let $h(\mathbf{x})$ be any nonconstant function that satisfies $\mathbf{B}_*\cdot\nabla h=0$, then the constant-$h$ surfaces are the magnetic surfaces of $\mathbf{B}_*$. If one writes, $h=h_0(\psi)+\delta h$, then to first order in the difference between $\mathbf{B}_*$ and $\mathbf{B}$ the deviation δh is given by $\mathbf{B}\cdot\nabla\delta h=-(\mathbf{B}_*-\mathbf{B})\cdot\nabla h_0(\psi)$. Using Eq. (7) to rewrite $\mathbf{B}\cdot\nabla\delta h$, one finds

$$\left(\frac{\partial}{\partial\varphi}+\iota\frac{\partial}{\partial\theta}\right)\delta h=-\frac{\mathbf{B}_*\cdot\nabla\psi}{\mathbf{B}\cdot\nabla\varphi}\frac{dh_0}{d\psi}. \tag{8}$$

The dominant effect on the loss of $\mathbf{B}_*$ surfaces is resonances between terms in the decomposition of $\mathbf{B}_*\cdot\nabla\psi/\mathbf{B}\cdot\nabla\varphi$ in a Fourier series in $e^{i(M\theta-N\varphi)}$ and the magnetic-field lines of the field $\mathbf{B}$. The resonances occur at the rational surfaces where $\iota(\psi)=N/M$. Near resonances one can write $\mathbf{B}_*\cdot\nabla\psi=-(m_e/e)(\nabla\times\boldsymbol{v}_\parallel)\cdot\nabla\psi$. To show this, note that $\mathbf{B}_*\cdot\nabla\psi=-(m_e/e)(\nabla\times\boldsymbol{v})\cdot\nabla\psi$, and $(\nabla\times\boldsymbol{v})\cdot\nabla\psi=\nabla\times(\boldsymbol{v}\times\nabla\psi)$, but $\boldsymbol{v}\times\nabla\psi=(v_\parallel/B)(\mathbf{B}\times\nabla\psi)-(\nabla\Phi_*\cdot\nabla\psi)\times(\mathbf{B}_*/B_*^2)+(\mathbf{B}_*\cdot\nabla\psi/B_*^2)\nabla\Phi_*$. Consequently, keeping only the lowest-order terms involving the electron density $(\nabla\times\boldsymbol{v})\cdot\nabla\psi=(\nabla\times\boldsymbol{v}_\parallel)\cdot\nabla\psi-\mathbf{B}\cdot\nabla(\nabla\Phi\cdot\nabla\psi/B^2)$. The term $\mathbf{B}\cdot\nabla(\nabla\Phi\cdot\nabla\psi/B^2)$ automatically vanishes at resonances, so the only term one needs to retain is $(\nabla\times\boldsymbol{v}_\parallel)\cdot\nabla\psi$.

Equation (8) for δh can be rewritten using three results: first, $\mathbf{B}_*\cdot\nabla\psi=-(m_e/e)(\nabla\times\boldsymbol{v}_\parallel)\cdot\nabla\psi$; second, $(\nabla\times\boldsymbol{v}_\parallel)\cdot\nabla\psi=(\mathbf{B}\times\nabla\psi)\cdot\nabla(v_\parallel/B)$; third, the θ and φ angles in the magnetic-field representation of Eq. (7) can be chosen so

$$2\pi\mathbf{B}=\mu_0(G\,\nabla\,\varphi+I\,\nabla\,\theta), \tag{9}$$

for a locally curl-free magnetic field. G is the current producing the toroidal magnetic field, and I is the toroidal current enclosed by the magnetic surfaces.[4] The rewritten Eq. (8) is

$$\left(\frac{\partial}{\partial\varphi}+\iota\frac{\partial}{\partial\theta}\right)\delta h=\frac{m_e}{e}\mu_0\left(G\frac{\partial v_\parallel/B}{\partial\theta}-I\frac{\partial v_\parallel/B}{\partial\varphi}\right)\frac{dh_0}{d\psi}. \tag{10}$$

In stellarator confinement $I=0$, and in axisymmetric systems $\partial/\partial\varphi=0$; the term involving I also does not contribute. For algebraic simplicity, we will ignore terms involving I in the remainder of the paper.

The parallel flow is determined by the condition that $\nabla\cdot(n\boldsymbol{v})=0$, which implies $\mathbf{B}\cdot\nabla(nv_\parallel/B)=-\nabla\cdot(n\boldsymbol{v}_\perp)$. When $n\ll n_B$, the divergence of the perpendicular flow is given by $\nabla\cdot(n\boldsymbol{v}_\perp)=\mathbf{B}\times\nabla\Phi\cdot\nabla(n/B^2)$. The electric potential is a function of ψ alone when $\Phi\gg T/e$, which is equivalent to the electron plasma having many Debye lengths, then

$$\left(\frac{\partial}{\partial\varphi}+\iota\frac{\partial}{\partial\theta}\right)\frac{nv_\parallel}{B}=-\mu_0 G\frac{\partial(n/B^2)}{\partial\theta}\frac{d\Phi}{d\psi}. \tag{11}$$

Variations in the geometry, which cause the electron density n to vary on the magnetic surfaces,[5] and variations in the magnetic-field strength on the magnetic surfaces can both drive resonant Fourier terms in $(\nabla\times\boldsymbol{v}_\parallel)\cdot\nabla\psi$ that are proportional to n/n_B. These terms cause a loss of confinement when they are sufficiently large to destroy the surfaces of the effective magnetic field $\mathbf{B}_*$.

Equations (10) and (11) can be combined into a single equation for the perturbation δh:

$$\left(\frac{\partial}{\partial\varphi}+\iota\frac{\partial}{\partial\theta}\right)^2\delta h=-\frac{m_e}{e}(\mu_0 G)^2\frac{\partial^2(n/B^2)}{\partial\theta^2}\frac{d\Phi}{dh_0}\left(\frac{dh_0}{d\psi}\right)^2. \tag{12}$$

The potential Φ, which in the unperturbed state is a function of ψ alone, has been written as $\Phi[h_0(\psi)]$.

In axisymmetric pure electron plasmas,[2] Eq. (12) for the perturbation to the lines of the $\mathbf{B}_*$ field is particularly simple. Letting $h_0=\psi$ and $\delta h=\delta\psi$, Eq. (12) implies

$$\delta\psi=-\frac{m_e}{e}\left(\frac{\mu_0 G}{\iota B}\right)^2\frac{d\Phi}{d\psi}+\text{const.} \tag{13}$$

The electric potential and the flux can be approximated by their cylindrical forms, $\Phi=(en/4\epsilon_0)r^2$ and $\psi=B_0\pi r^2$, where r is the minor radius of the torus and B_0 is a typical magnetic-field strength. This approximation yields

$$\frac{d\Phi}{d\psi}=\frac{en}{4\pi\epsilon_0 B_0}. \tag{14}$$

The variation of the field strength can be approximated as $(B_0/B)^2=1+2x/R_0$ with $x\equiv R-R_0$, which is the difference between the local major radius, R, and the major radius of the magnetic axis, R_0. One can also write $\mu_0 G=2\pi R_0 B_0$. Letting $\psi_*=\psi+\delta\psi$, one has $\psi_*=\pi B_0\{(x-x_s)^2+z^2\}$ where the outward shift of the magnetic axis of the $\mathbf{B}_*$ field relative to the axis of the $\mathbf{B}$ field is given by

$$x_s=\frac{R_0}{2\iota^2}\frac{n}{n_B}. \tag{15}$$

If one takes the equilibrium limit to be when the axis shift is half the minor radius a then the limiting density is $n/n_B=\iota^2 a/R_0$.

To calculate the perturbation in a nonaxisymmetric plasma, define the perturbation amplitudes δ_{MN} of the asymmetry by

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{n}{\langle n\rangle}\frac{B_0^2}{B^2}\right)=\sum_{MN}M^2\delta_{MN}\cos(N\varphi-M\theta). \tag{16}$$

$\langle n\rangle$ is the average density on the ψ surface, and B_0 is a typical magnetic field. Linearity implies that one perturbing coefficient δ_{MN} can be considered at a time. To obtain a nonsingular perturbation equation, let $h_0(\psi)=(\psi-\psi_{MN})^2/2$, where $\iota(\psi_{MN})=M/N$. Then,

$$\delta h=\frac{m_e}{e}\left(\frac{\mu_0 G}{\iota'}\right)^2\frac{\langle n\rangle}{B_0^2}\delta_{MN}\frac{d\Phi}{dh_0}\cos(N\varphi-M\theta), \tag{17}$$

where $\iota'\equiv d\iota/d\psi$ evaluated at ψ_{MN}.

The derivative of the potential $d\Phi/dh_0$ can be expressed in terms of the average electron density $\langle n\rangle$ on the rational surface ψ_{MN}. The electron density is given by Poisson's equation, $en/\epsilon_0=\nabla^2\Phi$, where $\nabla^2\Phi(h_0)=(d^2\Phi/dh_0^2)(\nabla h_0)^2+(d\Phi/dh_0)\nabla^2 h_0$. As $\psi\to\psi_{MN}$, one finds that $\nabla^2\Phi=(d\Phi/dh_0)(\nabla\psi)^2$. The typical field strength is defined by $B_0\equiv\mu_0 G/(2\pi R_0)$ with R_0 the major radius of the magnetic axis of $\mathbf{B}$ (the curve $\psi=0$), the average density by $\langle n\rangle$

$\equiv(\epsilon_0/e)(d\Phi/dh_0)\langle(\nabla\psi)^2\rangle$, and the radius squared of the resonant rational surface by $a^2\equiv\langle(\nabla\psi)^2\rangle/(2\pi B_0)^2$.

Using the expressions of the last paragraph, δh can be written as $4\delta h=\Delta^2\cos(N\varphi-M\theta)$, where

$$\Delta^2=2\frac{\delta_{MN}}{\iota'^2}\left(\frac{R_0}{a}\right)^2\frac{\langle n\rangle}{n_B}. \tag{18}$$

The equation for the perturbed surfaces of $\mathbf{B}_*$, which is $h=h_0+\delta h=$const, then yields

$$\psi-\psi_{MN}=\frac{s}{|s|}\sqrt{\Delta^2\left\{s^2-\sin^2\left(\frac{N\varphi-M\theta}{2}\right)\right\}}. \tag{19}$$

The surfaces of the $\mathbf{B}_*$ field are labeled by the constant s, and the identity $\cos(\alpha)=1-2\sin^2(\alpha/2)$ was used. The quantity Δ is the half-width of the magnetic islands in the $\mathbf{B}_*$ field. The $\mathbf{B}_*$ magnetic surfaces are lost when the islands from adjacent rational surfaces overlap.[6] The difference in the rotational transform between neighboring rational surfaces, $\delta\iota$, is $\delta\iota=(N/M)-N/(M+1)=(N/M)/(M+1)\approx\iota/M$. Island overlap occurs when $\delta\iota=2\iota'\Delta$. Consequently, equilibrium is lost due to the loss of surfaces of the $\mathbf{B}_*$ field when the average density $\langle n\rangle$ on the magnetic surfaces satisfies

$$\frac{\langle n\rangle}{n_B}\approx\frac{\iota^2}{8M^2}\frac{1}{\delta_{MN}}\left(\frac{a}{R_0}\right)^2. \tag{20}$$

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