Stability of pure electron plasmas on magnetic surfaces

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(Received 24 May 2004; accepted 8 July 2004; published online 20 September 2004)

The electrostatic analog of magnetohydrodynamic stability for a pure electron plasma on magnetic surfaces is examined. Perturbations that conserve the number of particles and the entropy of the plasma as well as maintain force balance and a temperature that is a spatial constant along the magnetic field lines are considered. It is shown that such perturbations require positive external energy and are, therefore, stable. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1789160]

I. INTRODUCTION

Toroidal plasmas with magnetic surfaces have been used for about half a century to confine quasineutral plasmas. Proposals to confine non-neutral plasmas in either axisymmetric levitated ring[1] or in stellarator[2] toroidal systems with magnetic surfaces are far more recent. The theory of pure electron plasmas confined on magnetic surfaces differs significantly from that of quasineutral plasmas. For example, the equilibrium of quasineutral plasmas is primarily determined by solving force balance, $\vec{\nabla}p=\vec{j}\times\vec{B}$ with $\vec{\nabla}\times\vec{B}=\mu_0\vec{j}$. The electric field is determined by making the transport ambipolar, and the charge imbalance is given by $\rho=\epsilon_0\vec{\nabla}\cdot\vec{E}$. In pure electron plasmas, the plasma density is sufficiently low so that its current has a negligible effect on the magnetic field. Force balance and rapid heat transport along the magnetic field lines determine[2] the form of the electron density, Eq. (8). The primary equilibrium equation is the equation for the electric potential $\nabla^2\Phi=-\rho/\epsilon_0$ with $\vec{E}=-\vec{\nabla}\Phi$.

As with research on any plasma confinement concept, the primary theoretical issues for pure electron confinement on magnetic surfaces are equilibrium, stability, and transport. Fundamental equilibrium theory is developed in Refs. 2 and 3. A heuristic discussion of transport is given in Ref. 2. Here the theory of plasma stability is developed for perturbations that are of sufficiently low frequency to preserve force balance and the equality of the electron temperature along the magnetic field lines. These two constraints are shown to imply that an otherwise ideal plasma is stable to all perturbations of the electric potential $\Phi(\vec{x})$. The source of this robust stability is that these two constraints eliminate plasma flows that cross the magnetic surfaces. Since the magnetic surfaces are rigid, the elimination of plasma flows across the surfaces means the plasma cannot effectively tap the enormous free energy that exists in the repulsive electrostatic potential.

The stability of a plasma confined in a torus was studied by O'Neil and Smith[4] with the assumptions of perfect toroidal symmetry and a purely toroidal magnetic field, which means without magnetic surfaces. The energy variation that they used to examine stability can be placed in the same form as the energy variation that is derived here. However, the nature of the solutions is different because a plasma confined on magnetic surfaces obeys different constraints.

II. PERTURBATION ENERGY

The change in the energy that occurs when a pure electron plasma is perturbed can be subtle to derive. A simple derivation, which is of interest in its own right, is based on the energy that is required to externally perturb the electric potential. The perturbations are produced by changing the charge distribution on a thin insulated toroidal shell, or grid, that lies on any surface, inside or outside of the plasma, on which the unperturbed electric potential is a constant, Φ_s. The external power that must be supplied to the shell to create a perturbation is the integral over the surface of the shell of the electric potential Φ times the time derivative of the surface charge σ

$$P_x=\oint\Phi\frac{\partial\sigma}{\partial t}da. \tag{1}$$

The derivation of Eq. (1) begins with the general expression for the power put into electromagnetic fields, $P_x=-\int_s\vec{j}\cdot\vec{E}d^3x$, which is a volume integral over the shell. One then uses $\vec{j}\cdot\vec{E}=-\vec{\nabla}\cdot(\Phi\vec{j})+\Phi\vec{\nabla}\cdot\vec{j}$, the continuity equation, $\vec{\nabla}\cdot\vec{j}=-\partial\rho/\partial t$, and $\vec{j}\cdot\hat{n}=0$ on the surface of the insulated shell. Finally, Poisson's equation, $\nabla^2\Phi=-\rho/\epsilon_0$, implies that no jump in the potential can occur across the thin shell, $[\Phi]=0$ and that $[\hat{n}\cdot\vec{\nabla}\Phi]=-\sigma/\epsilon_0$ with σ the surface charge density on the shell; within the shell $\rho d^3x=\sigma da$.

The electric potential on the shell can be written as $\Phi=\Phi_s+\delta\Phi$ with Φ_s the constant potential of the equilibrium state. The external power is $P_x=\Phi_s dQ_s/dt+\oint\delta\Phi(\partial\delta\sigma/\partial t)da$ with Q_s the total charge on the shell. The term $\Phi_s dQ_s/dt$ is not relevant to stability and can be eliminated by assuming either that the shell is grounded, $\Phi_s=0$, or that the total charge on the shell is not changed, $dQ_s/dt=0$.

Let $f_i(\theta,\varphi)$ be a set of dimensionless functions on the toroidal surface formed by the shell with θ and φ a poloidal and a toroidal angle on that surface. These functions are assumed to be orthonormal, $\oint f_i f_j^* w da$ with $w(\theta,\varphi)>0$ an arbitrary weight function. Changes in the electric potential on the shell can be written as

$$\delta\Phi=\sum_i\delta V_i(t)f_i^*(\theta,\varphi). \tag{2}$$

Similarly, the changes in the charge density on the shell can be written as

$$\delta\sigma = w\sum_i \delta Q_i(t) f_i^*(\theta,\varphi), \tag{3}$$

where δQ_i have units of charge. The required exernal power is then $P_x=\Sigma\, \delta V_i^* d\delta Q_i/dt$.

If the change in the plasma equilibrium is sufficiently small and slow, the change in the equilibrium associated with a potential change is linear and of the form

$$\delta Q_i = \sum_j C_{ij}\delta V_j, \tag{4}$$

where C_{ij} is a matrix that gives the effective capacitance of the shell. The capacitance matrix $\overleftrightarrow{C}$ can be determined empirically by measuring the change in the charge on the various pieces of a segmented shell as the potentials are varied. The required external power is $P_x=\delta\vec{V}^\dagger\cdot\overleftrightarrow{C}\cdot d\delta\vec{V}/dt$.

In principle the capacitance matrix need not be Hermitian, but we will show for the plasma model that we are considering that it is $\overleftrightarrow{C}=\overleftrightarrow{C}^\dagger$. When the capacitance is Hermitian, $P_x=d\delta W/dt$ with

$$\delta W = \tfrac{1}{2}\delta\vec{V}^\dagger\cdot\overleftrightarrow{C}\cdot\delta\vec{V} = \tfrac{1}{2}\oint \delta\Phi\,\delta\sigma da. \tag{5}$$

The surface charge on the shell is $\delta\sigma=-\epsilon_0[\hat{n}\cdot\vec{\nabla}\delta\Phi]$, which means one can convert the area integral over the shell of Eq. (5) into a volume integral over all of space except the volume occupied by the shell. That is,

$$\delta W = \tfrac{1}{2}\int [\epsilon_0(\vec{\nabla}\delta\Phi)^2 - \delta\rho\delta\Phi]d^3x, \tag{6}$$

where $\delta\rho=-\epsilon_0\nabla^2\delta\Phi$ is the perturbed charge density. Equation (6) is the equation for the change in energy associated with a perturbation. If $\delta W>0$, the system is stable to the perturbation. The expression of O'Neil and Smith[4] for δW, their Eq. (11), can be placed in the form of Eq. (6) with the use of their Eq. (12).

III. PLASMA CONSTRAINTS

The only difficulty in applying Eq. (6) is in determining how the charge density $\delta\rho$ varies as the electric potential $\delta\Phi$ is varied. The interpretation is that ρ is a functional of the potential Φ. In other words the change in the charge density $\delta\rho(\vec{x},t)$ is assumed to be calculable if the change in the potential $\delta\Phi(\vec{x},t)$ is known, but the relation between $\delta\rho$ and $\delta\Phi$ is more complicated than that of a function.

The calculation of $\delta\rho$ given $\delta\Phi$ can be carried out using four constraints. Two of these constraints are the constraints of an ideal plasma: the number of electrons $\int nd^3x$ and the entropy $\int sd^3x$ are conserved. The entropy per unit volume is $s=n\ln(c_sT^{3/2}/n)$ with c_s a constant. The third constraint is that the magnetic field provides force balance,

$$\vec{\nabla}p + en(\vec{E}+\vec{v}\times\vec{B}) = 0, \tag{7}$$

and the fourth constraint is $\vec{B}\cdot\vec{\nabla}T=0$, which comes from the rapid electron heat transport along the magnetic field lines. The notation used in the force balance equation is that p is the electron pressure with $p=nT$, n is the number of electrons per unit volume so the charge density is $\rho=-en$, and $\vec{v}$ is the electron fluid velocity.

The actual equilibrium equation has a term, $m_en\vec{v}\cdot\vec{\nabla}\vec{v}$, in addition to those given in Eq. (7). This term sets an upper limit on the density in pure electron plasmas, the Brillouin limit,[5,6] $n_B\equiv\epsilon_0B^2/2m_e$. The cause of the Brillouin limit is easily understood. The term $en\vec{v}\times\vec{B}$ must balance the Coulomb repulsion of the electrons. As the electron density becomes larger $|\vec{v}|$ must become larger. Eventually the repulsion of the centrifugal force associated with the term $m_en\vec{v}\cdot\vec{\nabla}\vec{v}$ becomes comparable with the electrostatic repulsion. A higher density makes the repulsive terms dominate and prevents an equilibrium. As the Brillouin limit is approached, $n\to n_B$ with $n<n_B$, stability is presumably lost even before equilibrium. However, the approach to the Brillouin limit is beyond the scope of this paper. Indeed, for the magnetic surfaces to play a role in the physics one must satisfy an even stricter limit on the density than $n\ll n_B$. The existence of magnetic surfaces is only relevant if the electrons can move faster by going along the magnetic field than across at the velocity $\vec{E}\times\vec{B}/B^2$. This constraint requires that $n/n_B\ll(\iota\lambda_d/R)^2$. The typical distance along a magnetic field line between points within a magnetic surface is R/ι, and $\lambda_d^2\equiv\epsilon_0T/ne^2$ defines the Deybe length λ_d. The distance R is the major radius of the torus and ι is the rotational transform. A magnetic field line advances on average $2\pi\iota$ radians poloidally each time it makes a toroidal circuit of the torus. A plasma is normally defined by the short Debye length limit, $\lambda_d\ll a$, where a is the minor radius of the torus, so in the plasma limit the density must be very small compared to the Brillouin limit for the magnetic surfaces to be important. When the limits $n/n_B\ll(\iota\lambda_d/R)^2\ll 1$ are satisfied, the term $m_en\vec{v}\cdot\vec{\nabla}\vec{v}$ is not only much smaller than $en\vec{E}$ but also much smaller than $\vec{\nabla}p$.

The analysis of this paper is carried out under the assumption of an isotropic plasma pressure. This approximation, although conventional, is not strictly correct for the instability of traditional interest for pure electron plasmas, the diocotron mode. This instability grows at a rate $\gamma\sim(E/B)/a$, where a is the minor radius of the torus. The time scale for electrons to precess poloidally due to their $\vec{E}\times\vec{B}/B^2$ velocity is $2\pi/\gamma$. For the pure electron plasmas of interest, the collision frequency is much smaller than γ, which means a perturbation can produce a small pressure anisotropy.

The imposition of the constraints due to the magnetic field is simplified by the use of (ψ,θ,φ) magnetic coordinates in which the magnetic field has simple forms. The magnetic field is curl free, so it can be written as $\vec{B}=\mu_0G_0\vec{\nabla}\varphi/2\pi$ with φ a dimensionless scalar potential for the magnetic field, $\nabla^2\varphi=0$. The quantity φ can be used as the toroidal angle. G_0 is a constant, the current that produces the magnetic field, $\mu_0G_0=\oint\vec{B}\cdot d\ell$. One can also write a magnetic field that forms surfaces as $\vec{B}=\vec{\nabla}\psi\times\vec{\nabla}\theta/2\pi+\iota(\psi)\vec{\nabla}\varphi\times\vec{\nabla}\psi/2\pi$, where θ is a magnetic poloidal angle. The toroidal magnetic flux enclosed by a magnetic surface is ψ, and $\iota(\psi)$

is the rotational transform. The existence of magnetic surfaces means each field line trajectory in a constant-ψ surface comes arbitrarily close to each point on that surface. This is true "almost everywhere" in the language of mathematics if $\iota(\psi)$ is a nonconstant function of ψ.

In the presence of magnetic surfaces, the condition $\vec{B}\cdot\vec{\nabla}T=0$ implies $T(\psi)$. The temperature is a constant on a magnetic surface and the pressure is $p=nT(\psi)$. Since $\vec{B}\cdot\vec{\nabla}\psi=0$, electron force balance along the magnetic field, Eq.(7), implies that when field lines cover constant-ψ surfaces

$$n(\psi,\Phi) = N(\psi)\exp\left(\frac{e\Phi}{T(\psi)}\right). \tag{8}$$

The electron flow velocity is also given by Eq. (7) and is

$$\vec{v} = \frac{v_\parallel}{B}\vec{B} + \frac{\vec{B}}{B^2}\times\left(\vec{\nabla}\Phi - \frac{T}{e}\vec{\nabla}\ln n - \frac{1}{e}\vec{\nabla}T\right). \tag{9}$$

The electron flow across the magnetic surfaces is zero,

$$\vec{v}\cdot\vec{\nabla}\psi = -2\pi\frac{\partial}{\partial\theta}\left(\Phi - \frac{T}{e}\ln n\right) = 0, \tag{10}$$

since n is given by (8).

The number density n and the entropy density s are carried by the flow in an ideal fluid. Consequently, the absence of a flow across the field lines in both the unperturbed and perturbed plasma states implies that the number of particles and the entropy must be conserved flux surface by flux surface. These constraints are $\delta\int n\mathcal{J}d\theta\, d\varphi=0$ and $\delta\int n\ln(T^{3/2}/n)\mathcal{J}d\theta\, d\varphi=0$ with the Jacobian of (ψ,θ,φ) coordinates $\mathcal{J}=\mu_0 G_0/(2\pi B)^2$. Using the surface average of a function $f(\vec{x})$,

$$\langle f\rangle \equiv \frac{\oint f\mathcal{J}d\theta\, d\varphi}{\oint \mathcal{J}d\theta\, d\varphi}, \tag{11}$$

the two constraints are $\delta\langle n\rangle=0$ and $\delta\langle n\ln(T^{3/2}/n)\rangle=0$. These two constraints determine how $N(\psi)$ and $T(\psi)$ change as Φ changes.

Equation (8) implies that the variation in the density is given by

$$\delta n = \{\delta N/N + \delta(e\Phi/T)\}n, \tag{12}$$

so the constraint $\delta\langle n\rangle=0$ is

$$\frac{\delta N}{N} = -\frac{\left\langle n\delta\frac{e\Phi}{T}\right\rangle}{\langle n\rangle}. \tag{13}$$

The perturbation to the temperature is given by the constraint on the entropy $\delta\langle n\ln(T^{3/2}/n)\rangle=0$. This constraint can be rewritten using $\langle\delta n\rangle=0$ and $\langle(\ln T^{3/2})\delta n\rangle=(\ln T^{3/2})\langle\delta n\rangle=0$ as

$$\delta\left\langle n\ln\left(\frac{T^{3/2}}{n}\right)\right\rangle = \frac{3}{2}\frac{\delta T}{T}\langle n\rangle - \langle(\ln n)\delta n\rangle. \tag{14}$$

Equation (8) plus $\langle(\ln N)\delta n\rangle=(\ln N)\langle\delta n\rangle=0$ imply $\langle(\ln n)\delta n\rangle=\langle(e\Phi/T)\delta n\rangle$. Equations (12) and (13) then imply

$$\frac{3}{2}\frac{\delta T}{T} = \frac{\langle n\rangle\left\langle n\frac{e\Phi}{T}\delta\frac{e\Phi}{T}\right\rangle - \left\langle n\frac{e\Phi}{T}\right\rangle\left\langle n\delta\frac{e\Phi}{T}\right\rangle}{\langle n\rangle^2}. \tag{15}$$

Let

$$\Delta \equiv \langle n\rangle\left\langle n\frac{e\Phi}{T}\delta\Phi\right\rangle - \left\langle n\frac{e\Phi}{T}\right\rangle\langle n\delta\Phi\rangle, \tag{16}$$

then

$$\frac{\delta T}{T} = \frac{e\Delta/T}{\frac{3}{2}\langle n\rangle^2 + \langle n\rangle\left\langle n\left(\frac{e\Phi}{T}\right)^2\right\rangle - \left\langle n\frac{e\Phi}{T}\right\rangle^2}. \tag{17}$$

The Hermiticity of the the capacitance matrix can be shown to follow if $\int\delta\rho_1\delta\Phi_2 d^3x=\int\delta\rho_2\delta\Phi_1 d^3x$ for any two perturbations, which is true if $\langle\delta n_1\delta\Phi_2\rangle=\langle\delta n_2\delta\Phi_1\rangle$ with $\delta\rho=-e\delta n$. Now

$$\frac{T}{e}\langle\delta\Phi_1\delta n_2\rangle = \langle n\delta\Phi_1\delta\Phi_2\rangle - \frac{\langle n\delta\Phi_1\rangle\langle n\delta\Phi_2\rangle}{\langle n\rangle} - \frac{\mathcal{T}_{1,2}}{\langle n\rangle}, \tag{18}$$

where the effect of the temperature change is

$$\mathcal{T}_{1,2} \equiv \frac{\Delta_1\Delta_2}{\frac{3}{2}\langle n\rangle^2 + \langle n\rangle\left\langle n\left(\frac{e\Phi}{T}\right)^2\right\rangle - \left\langle n\frac{e\Phi}{T}\right\rangle^2} \tag{19}$$

with $\Delta_1\equiv\langle n\rangle\langle n(e\Phi/T)\delta\Phi_1\rangle-\langle n(e\Phi/T)\rangle\langle n\delta\Phi_1\rangle$ and Δ_2 defined analogously. Clearly, $\langle\delta n_1\delta\Phi_2\rangle=\langle\delta n_2\delta\Phi_1\rangle$, which ensures the Hemiticity of the capacitance matrix.

IV. PERTURBED ENERGY ON A SURFACE

The absence of a plasma flow across the rigid magnetic surfaces, $\vec{v}\cdot\vec{\nabla}\psi=0$, makes it useful to examine the effect of the perturbation on the energy, magnetic surface by magnetic surface. The variation in the energy, (6), can be written as $\delta W=\int\delta\overline{w}(\psi)(d\mathcal{V}/d\psi)d\psi$, where

$$\delta\overline{w} \equiv \tfrac{1}{2}\langle\epsilon_0(\vec{\nabla}\delta\Phi)^2\rangle + \delta\overline{w}_p(\psi), \tag{20}$$

$\delta\overline{w}_p(\psi)\equiv(e/2)\langle\delta n\delta\Phi\rangle$ and the volume enclosed by a flux surface is $\mathcal{V}(\psi)=\int\mathcal{J}d\psi\, d\theta\, d\varphi$. Using (18) the plasma contribution can be written as

$$\delta\overline{w}_p = \frac{e^2}{2\langle n\rangle\mathbf{T}}\{(\langle n\rangle\langle n(\delta\Phi)^2\rangle - \langle n\delta\Phi\rangle^2) - \boldsymbol{\mathcal{T}}\}, \tag{21}$$

where the contribution to $\delta\overline{w}_p$ due to the variation in the temperature is given by $\boldsymbol{\mathcal{T}}=\boldsymbol{\mathcal{T}}_{1,2}$ with $\delta\Phi_1=\delta\Phi_2=\delta\Phi$. The term $\boldsymbol{\mathcal{T}}$ is always positive and therefore destabilizing.

The ratio of the plasma energy to the electric field energy is

$$\frac{2\delta\overline{w}_p}{\epsilon_0(\vec{\nabla}\,\delta\Phi)^2} \approx \frac{1}{(k\lambda_d)^2}, \tag{22}$$

where the square of the Debye length is $\lambda_d^2 \equiv \epsilon_0 T/ne^2$ and the characteristic wave number k satisfies $k|\delta\Phi| \approx |\vec{\nabla}\delta\Phi|$. Since the electric field energy is positive, instability is possible only if the Debye length is small compared to the size of the plasma. When the Debye length is small, the perturbed plasma energy is dominant, so any perturbations that would make $\delta\overline{w}_p(\psi)$ negative would also make $\delta\mathbf{W}$ negative. We will find that no such perturbations exist. Consequently, pure electron plasmas are stable to perturbations that obey the constraints of our analysis.

The unperturbed electric potential within the plasma can be written as $\Phi(\psi,\theta,\varphi)=\overline{\Phi}(\psi)+\phi T/e$, where ϕ is dimensionless and of order unity with $\langle n\phi\rangle=0$. Similarly the perturbed potential can be written as $\delta\Phi=\delta\overline{\Phi}(\psi,t)+(T/e)\delta\phi$ with $\langle n\delta\phi\rangle=0$. One then finds that

$$\delta\overline{w}_p = \frac{\langle n\rangle T}{2}\left(\frac{\langle n(\delta\phi)^2\rangle}{\langle n\rangle} - \frac{\langle n\phi\delta\phi\rangle^2}{\frac{3}{2}\langle n\rangle^2 + \langle n\rangle\langle n\phi^2\rangle}\right). \tag{23}$$

This expression is always greater than or equal to zero. To see this note the $3/2$ in $\delta\overline{w}_p(\psi)$ increases its magnitude, so

$$\delta\overline{w}_p \geqslant \frac{\mathbf{T}}{2\langle n\phi^2\rangle}(\langle n\phi^2\rangle\langle n(\delta\phi)^2\rangle - \langle n\phi\delta\phi\rangle^2). \tag{24}$$

Let $\phi=c_\phi f(\psi,\theta,\varphi)$ with f normalized so $\langle nf^2\rangle=\langle n\rangle$, and let $\delta\phi=\delta c_\phi f+\delta c_g g(\psi,\theta,\varphi)$ with g normalized so $\langle ng^2\rangle=\langle n\rangle$ and made unique by the choice $\langle nfg\rangle=0$. Then, $\delta\overline{w}_p \geqslant (\langle n\rangle T/2)\times(\delta c_g)^2$. If $\delta c_g=0$, then $\delta\overline{w}_p=(\langle n\rangle T/2)\delta c_\phi^2/(1+2c_\phi^2/3)$, so $\delta\overline{w}_p$ is positive for all perturbations of the potential.

Pure electron plasmas confined on magnetic surfaces are robustly stable to low frequency perturbations, which are the electrostatic analogs of the well-known magnetohydrodynamic instabilities of quasineutral plasmas confined by a magnetic field.

ACKNOWLEDGMENTS

The author would like to thank Thomas Sunn Pedersen for many useful discussions.

This work was supported by the NSF/DoE partnership in plasma science Grant No. Phy-03-17359.

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