# The effect of the electric field on the confinement of electron plasmas on magnetic surfaces

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The neoclassical confinement of electron plasmas on magnetic surfaces is examined. The large electric field of these plasmas has the beneficial effect of making the diffusion coefficient proportional to $E^{-2}$, because of the $E\times B$ drift within the magnetic surfaces. Diffusion driven by the electron density gradient is expected to give an approximate confinement time $\tau \propto (e\Delta\Phi/T)^2$. However, the primary drive for electron transport is not the density gradient, but the radial electric field. This changes the scaling to $\tau \propto e\Delta\Phi/T$. The net effect of the electric field is still beneficial, but it is less than previously thought. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2794323]

The transport of quasineutral plasmas confined on magnetic surfaces, specifically in stellarators, has long been a topic of interest.[1,2] Confinement of electron-rich non-neutral plasmas in traps[3,4] and purely toroidal magnetic fields[5] have also been previously investigated, but recently the confinement of these electron plasmas on magnetic surfaces has become a topic of theoretical[6–8] and experimental[9–12] interest as well. The large electric field ($e\Delta\Phi/T \gg 1$) of an electron plasma was expected to play a large, beneficial role in the confinement of electrons. The confinement time scaling was previously estimated to be $\tau \approx (e\Delta\Phi/T)^2/\nu_c$, where $\nu_c$ is the collision frequency of the electrons.[7] Since $e\Delta\Phi/T \approx (a/\lambda_D)^2$ in an electron plasma,[6] where $a$ is the plasma minor radius and $\lambda_D$ is the Debye length, the confinement time scaled as[6,7] $\tau \approx (a/\lambda_D)^4\nu_c$. Therefore, any electron plasma confined on magnetic surfaces that satisfied the plasma condition $\lambda_D \gg a$ was predicted to have very long confinement times.[9]

In the present paper we will show that the above scaling was overly optimistic and that confinement time should be proportional to $e\Delta\Phi/T$ rather than $(e\Delta\Phi/T)^2$ for an electron plasma. This is because a convective flux due to the outward push of the radial electric field dominates the transport. This effect of the radial electric field on electron flux was previously derived[5] for an electron plasma in a purely toroidal magnetic geometry. The present paper extends that result to the more general geometry of stellarator magnetic surfaces.

Using kinetic theory,[13] a relatively simple procedure for calculating transport uses the balance between transport and the collisional creation of entropy[14,15] [Eq. (3)]. The conservation of energy and particles imply the energy per unit volume obeys $\partial\epsilon/\partial t+\vec{\nabla}\cdot\vec{q}=0$ and the number of particles per unit volume obeys $\partial n/\partial t+\vec{\nabla}\cdot\vec{\Gamma}=0$, where $\vec{q}$ is the flux of energy and $\vec{\Gamma}$ is the flux of particles. The thermodynamic equation per unit volume, $d\epsilon=Tds+\mu_c dn$, where $\mu_c$ is the chemical potential, then gives

$$\frac{\partial s}{\partial t}+\vec{\nabla}\cdot\left(\frac{1}{T}\vec{q}-\frac{\mu_c}{T}\vec{\Gamma}\right)=\vec{q}\cdot\vec{\nabla}\frac{1}{T}-\vec{\Gamma}\cdot\vec{\nabla}\frac{\mu_c}{T}. \quad (1)$$

The divergence term represents an entropy flux, which neither creates nor destroys entropy. The two terms on the right-hand side, however, give the rate that entropy is created by collisions. Collisions increase the entropy per unit volume, i.e., $s=-\int f\ln(f)d^3v$, at the rate

$$\dot{s}_c=-\int \ln(f)\mathcal{C}(f)d^3v, \quad (2)$$

where $f(\vec{x},\vec{v})$ is the distribution function of the electrons and $\mathcal{C}(f)=df/dt$ is the collision operator. Therefore,

$$\vec{q}\cdot\vec{\nabla}\frac{1}{T}-\vec{\Gamma}\cdot\vec{\nabla}\frac{\mu_c}{T}=-\int \ln(f)\mathcal{C}(f)d^3v. \quad (3)$$

The collisional processes of primary importance for the electrons in an electron plasma with a small ion fraction[16] are with neutrals and with other electrons. The collisions with neutrals are essentially energy conserving because of the large electron-neutral mass ratio, but they do scatter the electrons in pitch $\lambda\equiv v_\parallel/v$, with each collision changing $\lambda$ by an amount comparable to unity. Collisions between electrons scatter both pitch and energy and the scattering is diffusive in nature. For both types of collisions, the collision operator vanishes when operating on a local Maxwellian distribution, $\mathcal{C}(f_M)=0$. For simplicity and also from experimental observations,[17] we assume the electron temperature is a spatial constant. The distribution function of the electrons is then close to a local Maxwellian,

$$f_M(\psi)=\exp\left(\frac{\mu_c(\psi)-H}{T}\right). \quad (4)$$

The chemical potential $\mu_c=c_\mu T-e\Phi+T\ln(n/T^{3/2})$, where $c_\mu$ is a constant, $\Phi$ is the electric potential, $H=mv^2/2-e\Phi$ is the energy of an electron, and $\psi$ is the toroidal magnetic flux enclosed by a magnetic surface. In thermodynamic equilibrium $\mu_c$ is a spatial constant. The actual distribution function is $f=f_M\exp(\hat{f})$. The deviation from a Maxwellian is small, i.e., $|\hat{f}| \ll 1$, when the confinement time is long compared to the collision time. This is assumed to be the case.

Equation (3) can be integrated over a magnetic surface. With the constant temperature assumption, and since $\mu_c$ depends only on $\psi$, it has the form

$$\Gamma_T \frac{d(\mu_c/T)}{d\psi} = \int \ln(f)\mathcal{C}(f)d^3v\mathcal{J}d\theta d\varphi. \tag{5}$$

The total particle flux crossing a magnetic surface is $\Gamma_T \equiv \oint \vec{\Gamma}\cdot d\vec{a}$. The coordinate system is $(\psi,\theta,\varphi)$, where $\theta$ and $\varphi$ are the poloidal and toroidal angles, respectively, $\mathcal{J}$ is the coordinate Jacobian, and $d\vec{a}=\vec{\nabla}\psi\mathcal{J}d\theta d\varphi$ is the area element.

The diffusive flux is proportional to the departure from thermodynamic equilibrium, and is written in terms of a diffusion coefficient,

$$\Gamma_T = -D\frac{d(\mu_c/T)}{d\psi}. \tag{6}$$

Using the definition of $\mu_c$, this is[18]

$$\Gamma_T = -D\left(\frac{1}{n}\frac{dn}{d\psi} - \frac{e}{T}\frac{d\Phi}{d\psi}\right). \tag{7}$$

An electron plasma with $e\Delta\Phi/T \gg 1$ has a large electric field, i.e., $d\Phi/d\psi$, and the second term dominates. Therefore, $\Gamma_T \approx D(e/T)d\Phi/d\psi$ and the electric field has a detrimental effect on transport.

The deviation of the electron distribution function from a local Maxwellian is obtained from two observations. First, the densities at which pure electron plasmas can be confined are sufficiently low that the mean free path of the electrons is much longer than the size of the plasma. Second, the collision-free distribution function of the electrons is constant along their trajectories. If $\delta(\vec{x},\vec{v})$ is the deviation in $\psi$ of the collisionless electron trajectories from their home $\psi$-surface, then the constancy of $f=f_M(\psi)\exp(\hat{f})$ along the trajectories implies

$$\hat{f} = -\frac{d\ln(f_M)}{d\psi}\delta(\vec{x},\vec{v}). \tag{8}$$

The $\psi$ derivative is at fixed particle energy since the energy is fixed along the trajectory, so $d\ln(f_M)/d\psi = d(\mu_c/T)/d\psi$ and

$$\hat{f} = -\frac{d(\mu_c/T)}{d\psi}\delta(\vec{x},\vec{v}). \tag{9}$$

The relation between entropy production and the diffusion coefficient can be used to obtain the relation between the diffusion coefficient $D$ and the deviations from the $\psi$ surfaces; i.e., $\delta$. To derive this relation, note that the conservation of particles and energy by the collision operator imply $\int \ln(f_M)\mathcal{C}(f)d^3v=0$. In addition, $\mathcal{C}(f_M)=0$. To lowest nonvanishing order in $\hat{f}$, the entropy creation $-\int \ln(f)\mathcal{C}(f)d^3v = -\int \hat{f}\mathcal{C}(f_M\hat{f})d^3v$. Equations (5), (6), and (9) then imply

$$D(\psi) = -\int \delta(\vec{x},\vec{v})\mathcal{C}(f_M\delta)d^3v\mathcal{J}d\theta d\varphi. \tag{10}$$

By an appropriate definition of the average, i.e., $\langle\cdots\rangle$, the diffusion coefficient can be written as

$$D = n\langle\delta^2\rangle\nu_c\frac{dV}{d\psi}, \tag{11}$$

where $\nu_c$ is the collision frequency and $dV/d\psi = \int\mathcal{J}d\theta d\varphi$ is the derivative of the volume enclosed by a constant $\psi$ surface $V(\psi)$, with respect to $\psi$.

The hard part of the transport calculation is the calculation of the trajectory deviations; i.e., $\delta(\vec{x},\vec{v})$. Although it is difficult to make an exact calculation, an estimate is not difficult. The energy or Hamiltonian, i.e., $H=mv^2-e\Phi$, can be rewritten using the magnetic moment $\mu_m$ as $H=mv_\parallel^2/2 + \mu_m B - e\Phi$. The magnetic moment of the electrons, i.e., $\mu_m = mv_\perp^2/2B$, should not be confused with the chemical potential $\mu_c$. Deeply trapped electrons have a small parallel velocity at a minimum of $\mu_m B - e\Phi$ along a magnetic field line. These electrons conserve energy $H$, and action $J\equiv\oint v_\parallel d\ell$. They drift from one field line to another keeping $\mu_m B - e\Phi$ constant while lying at a minimum of $\mu_m B - e\Phi$ along the field line on which they are located. The electric potential can be separated into two parts:[6,19] $\Phi=\Phi_0(\psi) + \tilde{\Phi}(\psi,\theta,\varphi)$. The part of the potential that is constant on the magnetic surfaces has a variation across the plasma that is very large compared to the temperature $\Delta\Phi_0(\psi)\approx(T/e) \times(a/\lambda_D)^2$. The part of the electric potential that varies on the surfaces $(\tilde{\Phi})$ is comparable to the electron temperature. Likewise, the field strength can be written as $B_0(\psi)+\tilde{B}(\psi,\theta,\varphi)$. Unlike the variation in the electric potential, $B_0(\psi)$ varies relatively little across the plasma, while $\tilde{B}\approx B_0(a/R)$, where $R$ is the major radius. For deeply trapped electrons, the electron potential dominates the drift trajectories across the magnetic field lines, so $\Phi$ is almost constant along the trajectory. Since $\Phi=\Phi_0(\psi)+\tilde{\Phi}$, where $\tilde{\Phi}\approx T/e$,

$$\delta \approx \frac{T}{e(d\Phi_0/d\psi)}. \tag{12}$$

Therefore, the large electric field of an electron plasma has the beneficial effect of ensuring that excursions from the magnetic surfaces are small.

Electrons for which $|\lambda|=|v_\parallel/v|$ is sufficiently large move more quickly around a magnetic surface by moving along the magnetic field lines than across the lines with the $E\times B$ drift. This is true provided $\iota v_\parallel/R \gg 2\pi d\Phi_0/d\psi$, where $\iota$ is the rotational transform. Since $\psi\approx\pi B_0 a^2$, this inequality is equivalent to $e\Delta\Phi_0/T \ll (\iota a/R)(a/\rho)$, where $\rho$ is the electron gyroradius. The inequality holds as long as the Debye length satisfies $\lambda_D > \sqrt{\rho R/\iota}$ and ensures the electrons with a sufficiently large magnitude of pitch $|\lambda|=|v_\parallel/v|$ stay close to their home magnetic surface. The dependence of the deviation on pitch, i.e., $\delta(\lambda)$, means that both electron-electron and electron-neutral collisions give rise to transport with $D$ given by Eq. (11) and with $\delta$ given by Eq. (12).

The confinement time is $\tau \equiv \int n d^3x / \Gamma_T$. The total flux of electrons across a $\psi$ surface is obtained from Eqs. (7), (11), and (12):

$$\Gamma_T = \nu_c n \frac{T}{e(d\Phi/d\psi)} \frac{dV}{d\psi}. \tag{13}$$

This particle flux is very similar to that derived by Crooks and O'Neil[5] for an electron plasma in a purely toroidal magnetic configuration, but here it is for a more general geometry. Therefore,

$$\tau \approx \frac{e\Delta\Phi}{T} \frac{1}{\nu_c} \approx \left(\frac{a}{\lambda_D}\right)^2 \frac{1}{\nu_c}. \tag{14}$$

This is numerically equivalent to having an ordinary spatial diffusion coefficient $D_s \approx \lambda_D^2 \nu_c$, where the radial electron flux is $\Gamma_r = -D_s dn/dr$. However, the actual electron transport is driven by the gradient in the electric potential and not the density. The net effect of the large electric field of an electron plasma is still beneficial, but it is less than previously thought by a factor of $e\Delta\Phi/T$.

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